LORENTZ COVARIANCE FROM DISCRETE SYMMETRIES Z_2 & Z_3

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Summary

We argue that the deep origin of the Lorentz covariance resides in symmetries acting on the quantum level. The anticommutation relations resulting from Pauli's exclusion principle engender the $SL(2, C)$ symmetry in the Hilbert space of two-fermion states, while cubic ternary generalization of the exclusion principle leads to a $Z3$ covering of the Lorentz group engendering also the $SU(3)$ color symmetry. We show how the $Z3$-Lorentz symmmetry intertwined with the $SU(3)$ symmetry group is implemented in a generalized Dirac equation for colored quarks.

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- **Space and time**
- **Material bodies**
- **Forces acting between them**

Fundamental relationship

- **Newton's third law of dynamics**

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{1}$$

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- **shows the relation between three different realms which are dominant in our perception and description of physical world: massive bodies (m), force fields responsible for interactions between the bodies ("F") and space-time relations defining the acceleration ("a").**

Three realms

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- The same three ingredients are found in physics of fundamental interactions: we speak of elementary particles and fields evolving in space and time.
- we deliberately wrote

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{2}$$

in order to separate the directly observable entity (**a**) from the product of two entities whose definition is much less direct and clear.

Causal relationship

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- Also, by putting the acceleration alone on the left-hand side, we underline the causal relationship between the phenomena: the force is the cause of acceleration, and not vice versa.
- In modern language, the notion of force is generally replaced by that of a field.
- The fact that the three ingredients are related by the equation (1) may suggest that perhaps only two of them are fundamentally independent, the third one being the consequence of the remaining two.

Three aspects

The three aspects of theories of fundamental interactions can be symbolized by three orthogonal axes, as shown in following figure, which displays also three choices of pairs of independent properties from which we are supposed to be able to derive the third one.

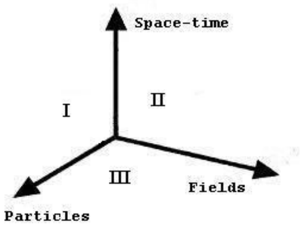

Figure: The three realms of Physics

Three types of theories

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Three types of theories

- The attempts to understand physics with only two realms out of three represented in (7) have a very long history. They may be divided in three categories, labeled *I*, *II* and *III* in the Figure.
- In the category *I* one can easily recognize Newtonian physics, presenting physical world as collection of material bodies (particles) evolving in absolute space and time, interacting at a distance. Newton considered light being made of tiny particles, too; the notion of fields was totally absent.

Three types of theories

- Theories belonging to the category *II* assume that physical world can be described uniquely as a collection of fields evolving in space-time manifold. This approach was advocated by **lord Kelvin**, **Einstein**, and later on by **Wheeler**.

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- **As a follower of Maxwell and Faraday, Einstein believed in the primary role of fields and tried to derive the equations of motion as characteristic behavior of singularities of fields evolving in space-time.**

Three types of theories

- **The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.**

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- The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.
- It is true that space-time coordinates cannot be treated on the same footing as conserved quantities such as energy and momentum; we often forget that they exist rather as bookkeeping devices, and treating them as real objects is a "bad habit", as pointed out by D. Mermin.

Three types of theories

- **Seen under this angle, the idea to derive the geometric properties of space-time, and perhaps its very existence, from fundamental symmetries and interactions proper to matter's most fundamental building blocks seems quite natural.**

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- Many of those properties do not require any mention of space and time on the quantum mechanical level, as was demonstrated by Born and Heisenberg in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras.
- The non-commutative geometry is another example of formulation of space-time relationships in purely algebraic terms.

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- **In other words, the Lorentz symmetry observed on the macroscopic level, acting on what we perceive as space-time variables, is an averaged version of the symmetry group acting in the Hilbert space of quantum states of fundamental particle systems.**

Of matter and space-time

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Of matter and space-time

- **Extending the Lorentz transformations to space and time coordinates modified also Newtonian mechanics so that it could become invariant under the Lorentz instead of the Galilei group.**
- **In the textbooks introducing the Lorentz and Poincaré groups the accent is put on the transformation properties of space and time coordinates, and the invariance of the Minkowskian metric tensor $g_{\mu\nu} = diag(+,-,-,-)$.**
- **But neither the components of $g_{\mu\nu}$, nor the space-time coordinates can be given an intrinsic physical meaning; they are not related to any conserved or directly observable quantities, contrary to ω and $\mathbf{k}$.**

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- **All acts of observation to which we have access reduce at the end of the experimental chain to PHOTONS interacting with ELECTRONS (or other leptons, or baryons).**
- **No wonder that the dual space to the space of four-vectors like $k^\mu = [\frac{\omega}{c}, \mathbf{k}]$ or $j^\mu = [\rho c, \mathbf{j}]$, i.e. the space of linear functionals on it, inherits the same symmetry group.**

Quantum covariance

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- **Quantum Mechanics started as a non-relativistic theory, but very soon its relativistic generalization was created.**
- **As a result, the wave functions in the Schroedinger picture were required to belong to one of the linear representations of the Lorentz group, which means that they must satisfy the following covariance principle:**

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

Quantum covariance

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- The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial...
- As in many other fundamental relations, the seemingly simple equation

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

creates a bridge between two totally different realms: the space-time accessible via classical macroscopic observations, and the Hilbert space of quantum states. It can be interpreted in two opposite ways, depending on which side we consider as the cause, and which one as the consequence.

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- or maybe it is already present as symmetry of quantum states, and then implemented and extended to the macroscopic world in classical limit ? In such a case, the covariance principle should be written as follows:
- $$\Lambda^{\mu'}_{\mu}(S) j^{\mu} = j^{\mu'}(\psi') = j^{\mu'}(S(\psi)),$$

  In the above formula
  $$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$
  is the Dirac current, ψ is the electron wave function.

- **In view of the analysis of the causal chain, it seems more appropriate to write the same transformations with Λ depending on S:**

$$\psi'(x^{\mu'}) = \psi'(\Lambda^{\mu'}_{\nu}(S)x^{\nu}) = S\psi(x^{\nu}) \tag{3}$$

$$\Lambda^{\mu'}_{\mu}(S)x^{\mu}(\psi, \bar{\psi}) = x^{\mu'}(S\psi, \bar{\psi}\bar{S}). \tag{4}$$

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- This form of the same relation suggests that the transition from one quantum state to another, represented by the unitary transformation S is the primary cause that implies the transformation of observed quantities such as the electric 4-current, and as a consequence, the apparent transformations of time and space intervals measured with classical physical devices.

Quantum covariance

- **Although mathematically the two formulations are equivalent, it seems more plausible that the Lorentz group resulting from the averaging of the action of the $SL(2, \mathbf{C})$ in the Hilbert space of states contains less information than the original double-valued representation which is a consequence of the particle-anti-particle symmetry, than the other way round.**

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- **In what follows, we shall draw physical consequences from this approach, concerning the strong interactions in the first place.**

First principles

- **Our questioning about the cause of measurable effects should not stop at the stage of *forces*, which are but expressions of effects of countless fundamental interactions, just like the thermodynamical pressure is in fact an averaged result of countless atomic collisions.**

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- On a classical level, when theory permits, the symbolical force can be replaced by a more explicit expression in which fields responsible for the forces do appear, like in the case of the Lorentz force.

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- $$A_\mu(\mathbf{r}, t) = \frac{1}{4\pi c} \int\int\int \frac{j_\mu(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{\mid \mathbf{r} - \mathbf{r}' \mid} d^3\mathbf{r}'. \qquad (5)$$

  then we get the field tensor given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

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- with $\psi^{\dagger} = \bar{\psi}^{T} \gamma^{5}, \quad \text{where}$

$$\gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- **In fact, the four-component complex function ψ is composed of two two-component spinors,** ξ_α and $\chi_{\dot\beta}$,

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- which are supposed to transform under two non-equivalent representations of the $SL(2, \mathbf{C})$ group:

$$\xi_{\alpha'} = S^{\alpha}_{\alpha'} \xi_\alpha, \quad \chi_{\dot\beta'} = S^{\dot\beta}_{\dot\beta'} \chi_{\dot\beta'}, \tag{7}$$

- **The electric charge conservation is equivalent to the annulation of the four-divergence of j^μ:**

$$\partial_\mu j^\mu = \left(\partial_\mu \psi^\dagger \gamma^\mu\right) \psi + \psi^\dagger \left(\gamma^\mu \partial_\mu \psi\right) = 0, \tag{8}$$

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- from which we infer that this condition will be satisfied if we have

$$\partial_\mu \psi^\dagger \gamma^\mu = -m\psi^\dagger \text{ and } \gamma^\mu \partial_\mu \psi = m\psi, \qquad (9)$$

which is the Dirac equation.

- **In terms of the spinorial components ξ and χ the Dirac equation can be seen as a pair of two coupled equations which can be written in terms of Pauli's σ-matrices:**

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- The relativistic invariance imposed on this equation is usually presented as follows: under a Lorentz transformation Λ the 4-current j^μ undergoes the following change:

$$j^\mu \to j^{\mu'} = \Lambda^{\mu'}_{\mu} j^\mu. \tag{11}$$

- **This means that the matrices γ^μ must transform as components of a 4-vector, too. Parallelly, the components of the bi-spinor ψ must be transformed in a way such as to leave the form of the Dirac equations unchanged: writing symbolically the transformation of $\mid \psi >$ as $\mid \psi' >= S \mid \psi >$, and $< \psi' \mid = < \psi \mid S^{-1}$, we should have**

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$$j^{\mu'} =< \psi' \mid \gamma^{\mu'} \mid \psi' >=< \psi \mid S^{-1} \gamma^{\mu'} S \mid \psi >= \Lambda^{\mu'}_{\mu} j^{\mu} = \Lambda^{\mu'}_{\mu} < \psi \mid \gamma^{\mu} \mid \psi > \tag{12}$$

from which we infer the transformation rules for gamma-matrices:

$$S^{-1} \gamma^{\mu'} S = \Lambda^{\mu'}_{\mu} \gamma^{\mu}. \tag{13}$$

Pauli's exclusion principle

- **The exclusion principle for the fermions is one of the most important facts of the elementary particle physics. In its first formulation in** 1924 **by Wolfgang Pauli only the states of the electrons in atomic shells were concerned: two electrons with the same quantum numbers can coexist only if their extra quantum number related to the proper spin of the electron, takes on different values out of the only two accessible ones.**

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- **Not only does it explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse, as suggested by Ehrenfest, and proved later by Dyson in his seminal papers.**

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- **In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.**
- **The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the "bra" and "ket" vectors.**

Pauli's exclusion principle

Consider the wave function of a particle in the state $\mid x >$,

$$\Phi(x) = < \psi \mid x > . \tag{14}$$

A two-particle state of $(\mid x >, \mid y >)$ is a tensor product

$$\mid \psi > = \sum \Phi(x,y) \, (\mid x > \otimes \mid y >). \tag{15}$$

If the wave function $\Phi(x,y)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(x,y) = -\Phi(y,x), \tag{16}$$

then $\Phi(x,x) = 0$ and such states have vanishing probability.

Pauli's exclusion principle

Conversely, suppose that $\Phi(x,x)$ does vanish. This remains valid in any basis provided the new basis $\mid x' >, \mid y' >$ was obtained from the former one via unitary transformation.

Let us form an arbitrary state being a linear combination of $\mid x >$ and $\mid y >$,

$$\mid z >= \alpha \mid x > + \beta \mid y >, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z,z) = < \psi \mid (\alpha \mid x > + \beta \mid y >) \otimes (\alpha \mid x > + \beta \mid y >), \qquad (17)$$

Pauli's exclusion principle

- **which develops as follows:**

$$\alpha^2 < \psi \mid x, x > + \alpha\beta < \psi \mid x, y >$$

$$+ \beta\alpha < \psi \mid y, x > + \beta^2 < \psi \mid y, y > =$$

$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (18)$$

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- **Now, as $\Phi(x, x) = 0$ and $\Phi(y, y) = 0$, the sum of remaining two terms will vanish if and only if (16) is satisfied, i.e. if $\Phi(x, y)$ is anti-symmetric in its two arguments.**

Pauli's exclusion principle

- **After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations**

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0 \tag{19}$$

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where $\psi(x) \mid 0 >=\mid x >$.

- **This anti-symmetry is so fundamental that it should be also Lorentz-invariant, or rather act as a Lorentz symmetry generating principle.**

Quantum covariance

- **The Pauli exclusion principle gives a hint about how it might work. In its simplest version, it introduces an anti-symmetric form on the Hilbert space describing electron's states:**

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \ \ \alpha, \beta = 1, 2; \quad \epsilon^{12} = 1,$$

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- Now, if we require that Pauli's principle must apply independently of the choice of a basis in Hilbert space, i.e. that after a linear transformation we get

$$\epsilon^{\alpha'\beta'} = S^{\alpha'}_{\alpha} S^{\beta'}_{\beta} \epsilon^{\alpha\beta} = -\epsilon^{\beta'\alpha'}, \quad \epsilon^{1'2'} = 1,$$

More explicitly,

$$\epsilon^{\alpha'\beta'} = S^{\alpha'}_{\alpha} S^{\beta'}_{\beta} \epsilon^{\alpha\beta} = -\epsilon^{\beta'\alpha'}, \quad \epsilon^{1'2'} = 1,$$

yields

$$S^{1'}_1 S^{2'}_2 - S^{1'}_2 S^{2'}_1 = 1$$

which means that the matrix $S^{\alpha'}_{\alpha}$ must have the determinant equal to 1, which defines the $SL(2, \mathbf{C})$ group.

- **Its real dimension is 6 (an arbitrary complex 2×2 matrix depends on four complex parameters, or equivalently, on 8 real partameters; imposing that the determinant must be equal to 1 is equivalent to two real equations, the imaginary part of the determinant being equal to 0, and this leaves just 6 free real parameters.**

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- **It is also easy to prove that the Lie algebras of both $SL(2, \mathbf{C})$ and the Lorentz group do coincide, both satisfying commutation relations defined by the formulae (22).**

The spinor realization of *six* $D = 4$ **Lorentz algebra generators** is often represented by two sets containing three generators each, the generators of three independent **spatial rotations** J_i and the three generators of the **Lorentz "boosts"** K_l involving the space-time transformations along the three space axes, defined as follows:

$$J_i = \frac{i}{2}\,\epsilon_{ijk}\left[\gamma^j, \gamma^k\right], \quad K_l = \frac{1}{2}\,\left[\gamma_l, \gamma_0\right] \tag{20}$$

. The explicit form in terms of tensorial products of 2×2 **matrices** is then as follows:

$$J_l = -\frac{i}{2}\mathbb{1}_2 \otimes \sigma_l, \quad K_i = -\frac{1}{2}\,\sigma_1 \otimes \sigma_i. \tag{21}$$

The full set of relations defining the Lorentz algebra is given by the following equations:

$$
\begin{aligned}
&[J_i, J_k] = \epsilon_{ikl} J_l, \quad [J_i, K_k] = \epsilon_{ikl} K_l, \\
&[K_i, K_k] = -\epsilon_{ikl} J_l.
\end{aligned} \tag{22}
$$

- **The notion of *isospin* was introduced as early as in 1932 by W. Heisenberg who noticed that at high energies when the strong nuclear interacions prevail, proton and neutron behave almost as two states of the same particle (*"the nucleon"* as far as the electromagnetic forces, much weaker that the nuclear ones, can be neglected.**

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- **To acknowledge the new discrete degree of freedom, taking on two values only, one needs a two-component wave function, just like in the case of the half-integer spin of the electron.**

- **In the high energy limit, when neutron and proton become undistinguishable, their state vectors in the Hilbert space of states can be linearly superposed under the condition that the resulting state vector is normalized to 1:**

$$| \, p' >=| \, \alpha \, | \, p > +\beta \, | \, n >|= 1, \; < p' \, | \, p' >= 1 \text{ implies } \bar{\alpha}\alpha + \bar{\beta}\beta = 1, \tag{23}$$

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- Such transformations form a group which can be represented by matrices of the following special type:

$$M = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \text{with } \det M = 1. \tag{24}$$

This real three-dimensional Lie group is called $SU(2)$, and is topologically isomorphic with a 3-sphere.

The Lie algebra of the $SU(2)$ **group** is isomorphic with the Lie algebra $SO(3)$ of three-dimensional Euclidean rotations. Its lowest-dimensional representation is given by the following three 2×2 **matrices**:

$$I_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

The two states of the nucleon form a basis of the lowest-dimensional spin $\frac{1}{2}$ **representation** of the $SU(2)$ **group**. Two commuting operators of the $SU(2)$ Lie algebra are I^2 and I_3, **the Casimir operator**, and the third component of the isospin vector.

In the first model of isospin symmetry proposed by Heisenberg, the two states of a nucleon, proton and neutron, are represented by two "iso-spinors", the two-component columns as follows:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{26}$$

The third component of isospin I_3 acting on basic vectors produce the eigenvalues $+\frac{1}{2}$ for the proton and $-\frac{1}{2}$ for the neutron. A simple rule for the electric charge follows,

$$Q = I_3 + \frac{1}{2}, \tag{27}$$

attributing the electric charge $+1$ to the proton, and 0 to the neutron state, respectively.

It was also conjectured by **Yukawa** that strong interactions must be mediated by lighter particles called *mesons*, belonging to a three-dimensional representation of the $SU(2)$ **group**, intertwining between two isospinors belonging to the half-integer representation. The three intermediary particles, called later the **π-mesons**, can be represented by the following three-component columns, spanning the integer **isopin $= 1$ representation**:

$$\pi^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{28}$$

- **The eigenvalues of the I_3 operator on these states are $+1$, 0, and -1, respectively, coinciding with their electric charges. Therefore, the isospin 1 pion states satisfy the simple relation between the eigenvalues of electric charge and the third component of the isospin, $Q = I_3$. To produce a single formula valid for nucleons and pions alike, a new conserved quantity has to be introduced: the *hypercharge* Y.**

- The eigenvalues of the I_3 **operator** on these states are $+1$, 0, and -1, respectively, coinciding with their electric charges. Therefore, the **isospin** 1 **pion states** satisfy the simple relation between the eigenvalues of electric charge and the third component of the isospin, $Q = I_3$. To produce a single formula valid for nucleons and pions alike, a new conserved quantity has to be introduced: the *hypercharge* Y.
- Called also the *baryon number*, its eigenvalue is 1 for strongly interacting fermions (nucleons and hyperons discovered later), and 0 for mesons. As a result, one gets the **Gell-Mann-Okubo formula** relating the three quantum numbers characterizing strongly interacting particles:

$$Q = I_3 + \frac{Y}{2}. \tag{29}$$

In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$. Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.

Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d ("up" and "down"), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.

According to the QCD model, in stable bound state there is place for *two* quarks in the same u-state or d-state, but not three.

Quarks display quite unusual quatum numbers. Conservation and additivity of the baryon number lead to the conclusion that the baryon charge of quarks must be equal to $\frac{1}{3}$, that of anti-quarks being $-\frac{1}{3}$.
The half-integer isospin representation attributes the value $I_3 = +\frac{1}{2}$ to the u-quark (the "up" state) and the value $I_3 = -\frac{1}{2}$ to the d-quark (the "down" state).
Then, according to the Gell-Mann-Okubo formula, the electric charges of quarks are:

$$Q(u) = \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}, \quad Q(d) = -\frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = -\frac{1}{3}, \tag{30}$$

and obviously, $Q(\bar{u}) = -\frac{2}{3}, \quad Q(\bar{d}) = \frac{1}{3}$.

- **This suggests that a natural generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can form a stable configuration perceived as one of the strongly interacting particles (even the relatively short-lived resonances).**

- This suggests that a natural generalization of Pauli's exclusion principle would be that **no *three* quarks in the same state** can form a stable configuration perceived as one of the strongly interacting particles (even the relatively short-lived resonances).
- Now we need statistical properties that would allow the coexistence of two quarks with the same isospin, but not three. By analogy with the Fermi-Dirac statistics based on the Z_2 **nilpotent operators**, we should introduce the Z_3 **nilpotent operators** $\theta_A, \; A, B = 1, 2$, such that $\theta_A^2 \neq 0$, but $\theta_A^3 = 0$.

Let us require then the vanishing of wave functions representing the tensor product of *three* (but not necessarily two) identical states. That is, we require that $\Phi(x, x, x) = 0$ for any state $| \, x >$.
As in the former case, consider an arbitrary superposition of three different states, $| \, x >$, $| \, y >$ and $| \, z >$,

$$| \, w >= \alpha \, | \, x > + \beta \, | \, y > + \gamma \, | \, z >$$

and apply the same criterion, $\Phi(w, w, w) = 0$.

We get then, after developing the tensor products,

$$\Phi(w,w,w) = \alpha^3\Phi(x,x,x) + \beta^3\Phi(y,y,y) + \gamma^3\Phi(z,z,z)$$

$$+\alpha^2\beta[\Phi(x,x,y)+\Phi(x,y,x)+\Phi(y,x,x)]+\gamma\alpha^2[\Phi(x,x,z)+\Phi(x,z,x)+\Phi(z,x,x)]$$

$$+\alpha\beta^2[\Phi(y,y,x)+\Phi(y,x,y)+\Phi(x,y,y)]+\beta^2\gamma[\Phi(y,y,z)+\Phi(y,z,y)+\Phi(z,y,y)]$$

$$+\beta\gamma^2[\Phi(y,z,z)+\Phi(z,z,y)+\Phi(z,y,z)]+\gamma^2\alpha[\Phi(z,z,x)+\Phi(z,x,z)+\Phi(x,z,z)]$$

$$+\alpha\beta\gamma[\Phi(x,y,z)+\Phi(y,z,x)+\Phi(z,x,y)+\Phi(z,y,x)+\Phi(y,x,z)+\Phi(x,z,y)] = 0.$$

The terms $\Phi(x,x,x)$, $\Phi(y,y,y)$ and $\Phi(z,z,z)$ do vanish by virtue of the original assumption. Among the remaining terms, the combinations preceded by various independent powers of numerical coefficients α, β and γ, must vanish separately.

This is achieved if the following Z_3 symmetry is imposed on our wave functions:

$$\Phi(x,y,z) = j\,\Phi(y,z,x) = j^2\,\Phi(z,x,y).$$

with j representing the primitive root of 1,

$$j = e^{\frac{2\pi i}{3}}, \;\; j^3 = 1, \;\; j + j^2 + 1 = 0. \tag{31}$$

Note that the complex conjugates of functions $\Phi(x,y,z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x,y,z) = j^2\,\Psi(y,z,x) = j\,\Psi(z,x,y).$$

Let $A, B, ... = 1, 2, ..N$; we suppose that the generators θ^A span an associative algebra, in which the following cubic relations are imposed:

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B, \quad \text{with} \;\; j = e^{\frac{2\pi i}{3}} \tag{32}$$

We also introduce a similar set of *conjugate* generators $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, ... = 1, 2, ..., N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2\, \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j\, \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}, \tag{33}$$

We complete the constitutive relations by the commutation between the generators θ and their conjugates $\bar{\theta}$:

$$\theta^A \bar{\theta}^{\dot{B}} = -j \bar{\theta}^{\dot{B}} \theta^A. \tag{34}$$

The cubic analogue of binary anti-commutation relations defines a Z_3-graded generalization of Grassmann algebra $\mathcal{A}_{Z_3}$ The natural Z_3 grading attributes the Z_3-grade 1 to θ's and the Z_3-grade 2 to $\bar{\theta}$'s. Under the associative multiplication the grades add up modulo 3, so that e.g. the product $\theta^A \theta^B$ has the Z_3-grade 2, the product $\theta^A \bar{\theta}^{\dot{B}}$ has the Z_3-grade 0, etc.

The following two properties of the Z_3-graded analogue of Grassmann algebra are useful for modelling basic properties of quarks. First, it is easy to prove that only quadratic and cubic expressions do not vanish; all products of four and more generators must vanish. Indeed, let us consider a fourth-order expression, $\theta^A\theta^B\theta^C\theta^D$. We have, using the associativity property, the following identities:

$$\theta^A\theta^B\theta^C\theta^D = j\,\theta^B\theta^C\theta^A\theta^D = j^2\,\theta^B\theta^A\theta^D\theta^C = j^3\,\theta^A\theta^D\theta^B\theta^C = j^4\,\theta^A\theta^B\theta^C\theta^D, \tag{35}$$

and because $j^4 = j \neq 1$, the only solution is $\theta^A\theta^B\theta^C\theta^D = 0$.

Secondly, foreseeing that cubic combinations of quarks result in composite particles perceived as fermions (e.g. proton and neutron), the corresponding operators should anti-commute. This is true indeed, as the following simple computation shows:

$$(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}) = (-j)^3\bar{\theta}^{\dot{D}}(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}) = \dots$$

$$= (-j)^9(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}})(\theta^A\theta^B\theta^C) = -(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}})(\theta^A\theta^B\theta^C), \qquad (36)$$

The conjugate generators $\bar{\theta}^{\dot{B}}$ span an algebra $\bar{\mathcal{A}}$ isomorphic with $\mathcal{A}$.

With two quark states, u and d, we get only two independent cubic combinations and three independent quadratic combinations. Considering the formulae (32) and (33) with only two generators and replacing θ^1 by u and θ^2 by d, we see that the only non-vanishing cubic monomials are

$$udu = jduu = j^2 uud, \quad dud = judd = j^2 ddu, \quad \bar{u}\bar{d}\bar{u} = j^2 \bar{d}\bar{d}\bar{u} = j^2 = j\bar{u}\bar{u}\bar{d}, \tag{37}$$

corresponding to the proton and neutron, anti-proton and anti-neutron states. Due to the equation (35), all combinations of order four and higher do vanish identically. The only quadratic combinations satisfying relations (34) are as follows:

$$u\bar{d} = -j\bar{d}u, \quad \bar{u}d = -j^2 d\bar{u}, \quad u\bar{u} - d\bar{d} \tag{38}$$

corresponding to three pions , π^+, π^- and π^0.

By analogy with the derivation of the $SL(2, \mathbf{C})$ symmetry from the skew Z_2-symmetry of Pauli's exclusion principle, let us show that certain complex representation of the same symmetry group can be derived from the Z_3-graded generalization of the exclusion principle defined above. Let us suppose that as usual Dirac fermions, quark should be described by a four-component column composed of two Pauli-like spinors. According to the ternary Z_3-skew symmetric anti-commutation hypothesis, these spinors are operator-valued. Let us denote them in the usual way:

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \begin{pmatrix} d^1 \\ d^2 \end{pmatrix}, \quad \begin{pmatrix} \bar{u}^{\dot{1}} \\ \bar{u}^{\dot{2}} \end{pmatrix}, \quad , \quad \begin{pmatrix} \bar{d}^{\dot{1}} \\ \bar{d}^{\dot{2}} \end{pmatrix}. \tag{39}$$

where u^1 is the operator corresponding to the spin-up state of the u-quark, u^2 is the spin-down state of the u-quark, etc., with the conjugate operator valued spinors denoted by means of dotted indices: $\bar{u}^{\dot{1}}$, $\bar{u}^{\dot{2}}$, etc.

The cubic combinations supposed to produce the observable fermionic states will contain the mixed products of the components of both u and d quarks, which should also obey the **Pauli exclusion principle**. This means that although there can be two u or two d quarks out of three, they cannot display the same spin direction. Therefore the only possible products must be tthe following:

$$u^1 u^2 d^1, \quad u^1 u^2 d^2, \quad d^1 d^2 u^1, \quad d^1 d^2 u^2, \tag{40}$$

and of course the four similar expressions made of the conjugate quarks with dotted indices:

$$\bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{1}}, \quad \bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{2}}, \quad \bar{d}^{\dot{1}} \bar{d}^{\dot{2}} \bar{u}^{\dot{1}}, \quad \bar{d}^{\dot{1}} \bar{d}^{\dot{2}} \bar{u}^{\dot{2}}. \tag{41}$$

We notice that there are four independent cubic terms made of the combinations (uud) and $(\bar{u}\bar{u}\bar{d})$, and also four independent cubic expressions made of the combinations (udd) and $(\bar{u}\bar{d}\bar{d})$.

This is exactly the number of components one needs to describe two independent Dirac fermions, the proton and the neutron. This is so because the order in cubic products does not matter: according to the ternary anti-commutation laws (32, 33), all permutations are linearly dependent:

$$u^1 u^2 d^1 = j u^2 d^1 u^1 = j^2 d^1 u^1 u^2, \quad d^1 d^2 u^1 = j d^2 u^1 d^1 = j^2 u^1 d^1 d^2,$$
$$\bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{1}} = j^2 \bar{u}^{\dot{2}} \bar{d}^{\dot{1}} \bar{u}^{\dot{1}} = j \bar{d}^{\dot{1}} \bar{u}^{\dot{1}} \bar{u}^{\dot{2}}, \quad \bar{d}^{\dot{1}} \bar{d}^{\dot{2}} \bar{u}^{\dot{2}} = j^2 \bar{d}^{\dot{2}} \bar{u}^{\dot{2}} \bar{d}^{\dot{1}} = j \bar{u}^{\dot{2}} \bar{d}^{\dot{1}} \bar{d}^{\dot{2}}. \tag{42}$$

We can therefore produce a more symmetric expression containing all possible permutations as follows:

$$u^1u^2d^1 = \frac{1}{3}\left[u^1u^2d^1 + ju^2d^1u^1 + j^2d^1u^1u^2\right],$$

$$u^1d^2d^1 = \frac{1}{3}\left[u^1d^2d^1 + jd^2d^1u^1 + j^2d^1u^1d^2\right],$$

$$u^2u^1d^2 = \frac{1}{3}\left[u^2u^1d^2 + ju^1d^2u^2 + j^2d^2u^2u^1\right],$$

$$d^2d^1u^2 = \frac{1}{3}\left[d^2d^1u^2 + jd^1u^2d^2 + j^2u^2d^2d^1\right],$$

and

$$\bar{u}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{d}^{\dot{1}} = \frac{1}{3}\left[\bar{u}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{d}^{\dot{1}} + j^2\;\bar{u}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}} + j\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\right],$$

$$\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}} = \frac{1}{3}\left[\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}} + j^2\;\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}} + j\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\right],$$

$$\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}} = \frac{1}{3}\left[\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}} + j^2\;\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{u}^{\dot{2}} + j\;\bar{d}^{\dot{2}}\,\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\right],$$

$$\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}} = \frac{1}{3}\left[\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}} + j^2\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{d}^{\dot{2}} + j\;\bar{u}^{\dot{2}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\right].$$

This is similar to the "averaging" the product of two anti-commuting spinors using the skew-symmetric 2-form $\epsilon_{\alpha\beta}$: $\alpha, \beta = 1, 2,\ \epsilon_{12} = 1$. If the variables χ^α anti-commute, i.e. if $\chi^\alpha \chi^\beta = -\chi^\beta \chi^\alpha$, then we can write

$$\chi^1 \chi^2 = \frac{1}{2} \left[\epsilon_{\alpha\beta}\ \chi^\alpha \chi^\beta \right]. \tag{43}$$

By analogy with this Z_2 skew-symmetric case, we can introduce the Z_3 skew-symmetric 3-form playing a similar role; however, due to the existence of two independent combinations of indices, (121) and (212), two independent Z_3-skew symmetric 3-forms must be introduced, with an extra upper index labeling them. For the sake of generality, let us consider some abstract Z_3-graded algebra spanned by two generators $\theta^A,\ A, B, = 1, 2$ and their conjugates $\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}$, satisfying the set of constitutive relations (32, 33 and 34).

We introduce the following two 3-forms ρ^{α}_{ABC} and their conjugates $\bar{\rho}^{\dot\beta}_{\dot D\dot E\dot F}$, with $\alpha, \dot\beta = 1, 2$ and $A, \dot B = 1, 2$, whose components are given explicitly below:

$$\rho^1_{121} = 1, \ \rho^1_{211} = j^2, \ \rho^1_{112} = j, \quad \rho^2_{212} = 1, \ \rho^2_{122} = j^2, \ \rho^2_{221} = j. \tag{44}$$

$$\bar\rho^{\dot 1}_{\dot 1\dot 2\dot 1} = 1, \ \bar\rho^{\dot 1}_{\dot 2\dot 1\dot 1} = j, \ \bar\rho^{\dot 1}_{\dot 1\dot 1\dot 2} = j^2, \quad \bar\rho^{\dot 2}_{\dot 2\dot 1\dot 2} = 1, \ \bar\rho^{\dot 2}_{\dot 1\dot 2\dot 2} = j, \ \bar\rho^{\dot 2}_{\dot 2\dot 2\dot 1} = j^2, \tag{45}$$

Quite obviously, the 3-forms ρ^{α}_{ABC} and $\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}}$ satisfy the following Z_3-symmetry properties:

$$\rho^1_{121} = j^2\, \rho^1_{211} = j\, \rho^1_{112}, \quad \rho^2_{212} = j^2\, \rho^2_{122} = j\, \rho^2_{221}, \tag{46}$$

and

$$\bar{\rho}^{\dot{1}}_{\dot{1}\dot{2}\dot{1}} = j\, \bar{\rho}^{\dot{1}}_{\dot{2}\dot{1}\dot{1}} = j^2\, \bar{\rho}^{\dot{1}}_{\dot{1}\dot{1}\dot{2}} \quad \bar{\rho}^{\dot{2}}_{\dot{2}\dot{1}\dot{2}} = j\, \bar{\rho}^{\dot{2}}_{\dot{1}\dot{2}\dot{2}} = j^2\, \bar{\rho}^{\dot{2}}_{\dot{2}\dot{2}\dot{1}} \tag{47}$$

The constitutive ternary commutation formulae take on the following shortened form:

$$u^1 u^2 d^1 = \frac{1}{3}\, \rho^1_{ABC} u^A u^B d^C, \quad u^2 u^1 d^2 = \frac{1}{3}\, \rho^2_{ABC} u^A u^B d^C,$$

$$\bar{u}^{\dot 1} \bar{u}^{\dot 2} \bar{d}^{\dot 1} = \frac{1}{3}\, \bar{\rho}^{\dot 1}_{\dot A \dot B \dot C} \bar{u}^{\dot A} \bar{u}^{\dot B} \bar{d}^{\dot C}, \quad \bar{u}^{\dot 2} \bar{u}^{\dot 1} \bar{d}^{\dot 2} = \frac{1}{3}\, \bar{\rho}^{\dot 2}_{\dot A \dot B \dot C} \bar{u}^{\dot A} \bar{u}^{\dot B} \bar{d}^{\dot C}, \tag{48}$$

mapping the 12 degrees of freedom of the three quarks (u, u, d) **onto the four degrees of freedom of the proton, supposed to be a 1/2-spin fermion, and**

$$d^1 u^2 d^1 = \frac{1}{3}\, \rho^1_{ABC} d^A u^B d^C, \quad d^2 u^1 d^2 = \frac{1}{3}\, \rho^2_{ABC} u^A u^B d^C,$$

$$\bar{d}^{\dot 1} u^{\dot 2} d^{\dot 1} = \frac{1}{3}\, \bar{\rho}^{\dot 1}_{\dot A \dot B \dot C} d^{\dot A} u^{\dot B} d^{\dot C}, \quad \bar{d}^{\dot 2} u^{\dot 1} d^{\dot 2} = \frac{1}{3}\, \bar{\rho}^{\dot 2}_{\dot A \dot B \dot C} d^{\dot A} u^{\dot B} d^{\dot C} \tag{49}$$

mapping the 12 degrees of freedom of the three quarks (d, d, u) **onto the four degrees of freedom of the neutron.**

- **The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.**

- **The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.**
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- **The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.**
- **Let $U_A^{A'}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U_B^{B'} \, \theta^B$.**
- **We are looking for the solution of the covariance condition for the ρ-matrices:**

$$S_{\beta}^{\alpha'} \, \rho_{ABC}^{\beta} = U_A^{A'} \, U_B^{B'} \, U_C^{C'} \, \rho_{A'B'C'}^{\alpha'}. \tag{50}$$

Now, as $\rho^1_{121} = 1$, we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-**matrix** on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j.$$

This leads to the following equation:

$$S^{1'}_1 = U^{1'}_1 \, U^{2'}_2 \, U^{1'}_1 + j^2 \, U^{2'}_1 \, U^{1'}_2 \, U^{1'}_1 + j \, U^{1'}_1 \, U^{1'}_2 \, U^{2'}_1 = U^{1'}_1 \, (U^{2'}_2 \, U^{1'}_1 - U^{2'}_1 \, U^{1'}_2), \tag{51}$$

because $j^2 + j = -1$.

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$S^{2'}_1 = U^{2'}_1\, U^{1'}_2\, U^{2'}_1 + j^2\, U^{1'}_1\, U^{2'}_2\, U^{2'}_1 + j\, U^{2'}_1\, U^{2'}_2\, U^{1'}_1 = U^{2'}_1\,(U^{1'}_2\, U^{2'}_1 - U^{1'}_1\, U^{2'}_2), \tag{52}$$

The remaining two equations are obtained in a similar manner.

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears on the right-hand side.

$$S^{2'}_{1} = -U^{2'}_{1}\,[det(U)], \tag{53}$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$S^{1'}_{2} = -U^{1'}_{2}\,[det(U)], \qquad S^{2'}_{2} = U^{2'}_{2}\,[det(U)]. \tag{54}$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $S^{\alpha'}_{\beta}$ one gets immediately

$$\det(S) = [\det(U)]^3. \tag{55}$$

The matrices $S^{\alpha'}_{\beta}$ belong to the $SL(2,\mathbf{C})$ group if the resulting cubic combinations of quarks (proton and neutron) behave like Dirac spinors; therefore we have

$$\det(S^{\alpha'}_{\beta}) = 1. \tag{56}$$

This leads to

$$\left[\det(U^{A'}_{B})\right]^3 = 1, \tag{57}$$

which means that

$$\det(U) = 1,\ j,\ \text{or}\ j^2. \tag{58}$$

- We see that the invariance group of the Z_3-graded algebra of isospin quark components satisfying Z_3-symmetric cubic commutation relations is a Z_3-covering of the $SL(2, \mathbf{C})$ group. The Lie algebra of this group is a Z_3-covering of the Lorentz algebra.

- We see that the invariance group of the Z_3-**graded algebra** of isospin quark components satisfying Z_3-**symmetric cubic** commutation relations is a Z_3-**covering** of the $SL(2, \mathbf{C})$ **group**. The Lie algebra of this group is a Z_3-**covering** of the Lorentz algebra.
- There is another representation of Z_3 **symmetry** corresponding to an extra degree of freedom of quarks, the *colour*. Incorporating the colour, which is akin to three-valued spin due to its exclusivity - no quarks displaying the same colour can be present in a stable bound state - leads to a Z_3-**graded** generalization of the Dirac equation.

- **The discovery *hyperons*, new hadrons more massive than proton and neutron, forced introduction of a third quark s (from "strange") beyond the u and d. It led to the first successful application of the $SU(3)$ symmetry, acting on a three-component column uniting in a single vector the quark states u, d and s.**

- The discovery *hyperons*, new hadrons more massive than proton and neutron, forced introduction of a third quark s (from "strange") beyond the u and d. It led to the first successful application of the $SU(3)$ **symmetry**, acting on a three-component column uniting in a **single vector** the quark states u, d and s.
- The fundamental 3×3-**dimensional** representation of the $SU(3)$ **group** was supposed to act on the complex vector space spanned by three different quark states.

- The discovery *hyperons*, new hadrons more massive than proton and neutron, forced introduction of a third quark s (from "strange") beyond the u and d. It led to the first successful application of the $SU(3)$ symmetry, acting on a three-component column uniting in a single vector the quark states u, d and s.
- The fundamental 3×3-dimensional representation of the $SU(3)$ group was supposed to act on the complex vector space spanned by three different quark states.
- The observed groups of similar mesons and baryons were then conceived as irreducible representations produced by appropriate tensor products of two or three fundamental represenations.

The tensor product of two inequivalent fundamental representations, 3 and $\bar{3}$, decomposes into the sum of two irreducible representations, the 1-**dimensional** *singlet* and the 8-**dimensional** *octet*, while the tensor product of **three** fundamental representation decomposes into the sum of a singlet, two octets and one decuplet, according to the symbolic equations below:

$$3 \otimes \bar{3} = 1 + 8, \quad 3 \otimes 3 \otimes 3 = 1 + 8 + 8 + 10, \qquad (59)$$

The particle content of the baryon octet and decuplet is represented in the Figure below, with essential quantum numbers: hypercharge Y, electric charge Q, isospin I and strangeness S.

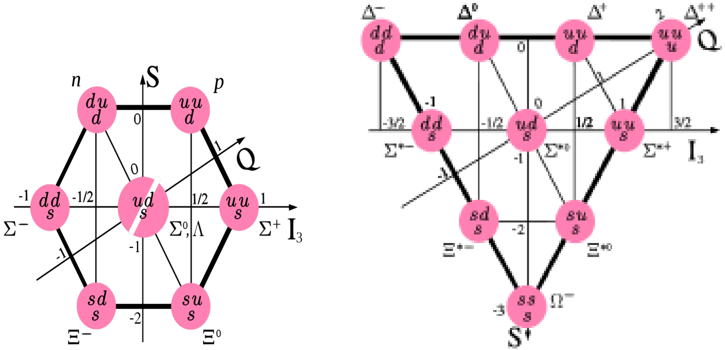

Figure: **Left: the baryon octet; Right: the baryon decuplet.**

The presence of THREE quarks of the same type, *ddd* (hyperon Δ^-) or *uuu* (hyperon Δ^{++}) which should be prohibited according to the isospin-statistics ternary generalization of Pauli's exclusion principle implemented by Z_3-skewsymmetric cubic commutation relations, suggested that there must be a new discrete degree of freedom for quarks, with THREE possible values. This new and exclusive degree of freedom was given the name of *colour*, with three values named by convention Red, Blue and Green.

- **In currently widely accepted Quantum Chromo-Dynamics (QCD) the extra color variable and the new symmetry it represents are taken into account by introducing three Dirac spinors, $\psi^A, \quad A = 1, 2, 3$, and the free Lagrangian is invariant under the action of the fundamental representation of the $SU(3)$ group:**

$$\psi^{B'} = U^{B'}_{A} \, \psi^A .$$

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$$\psi^{B'} = U^{B'}_{A}\, \psi^A.$$
- The action of the Lorentz group (identically on each of the Dirac spinors forming the color triplet) commutes with the action of the $SU(3)$ group.

Explicitly, the fundamental representation of the $SU(3)$ group acts on the following triplet of Dirac spinors:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{60}$$

The Lorentz group acts simultaneously on each of the "coloured" Dirac spinors via its standard 4-D spinorial representation

- **We shall extend the $Z_2 \times Z_2$ symmetry by Z_3 group, so that the system will mix not only the two spin $\frac{1}{2}$ states and particles with anti-particles, but the three colours as well.**

- We shall extend the $Z_2 \times Z_2$ symmetry by Z_3 **group**, so that the system will mix not only the two **spin** $\frac{1}{2}$ states and particles with anti-particles, but the three colours as well.
- The standard Dirac equation expressed in terms of two entangled Pauli spinors $\psi_{\pm}$ will be extended so as to incorporate six entangled Pauli spinors, to which three colours and three anti-colours are attributed.

- **The Z_3 symmetry can be combined with the Z_2 symmetry; 3 and 2 being prime numbers, the Cartesian product of the two is isomorphic with another cyclic group,**

$$Z_3 \times Z_2 = Z_6$$

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.

- The generalized Dirac equation is invariant under the discrete group $Z_3 \times Z_2 \times Z_2 \simeq Z_6 \times Z_2$ (which is not isomorphic with Z_{12} because 6, being divisible by 2 and by 3, is not a prime number).

The cyclic group Z_6 is represented in the complex plane by its generator $q = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$, and its powers from 1 to 6. In terms of the Z_3 group generated by j and Z_2 group generated by -1, we have $q = -j^2, \; q^2 = j, \; q^3 = -1, \; q^4 = j^2, \; q^5 = -j, \; q^6 = 1$, as shown in the figure (3) below.

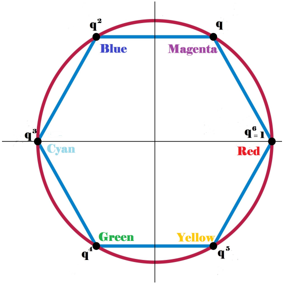

Figure: The six complex numbers q^k can be put into correspondence with three colours and three anti-colours.

- In analogy with colours labeling quark fields, if the "white" combination is represented by 0, then we have *two* linear colourless sums of three powers of q, namely

$$1 + q^2 + q^4 = 0 \text{ and } q + q^3 + q^5 = 0,$$

- In analogy with colours labeling quark fields, if the **"white"** combination is represented by 0, then we have *two* linear colourless sums of three powers of q, namely

$$1 + q^2 + q^4 = 0 \text{ and } q + q^3 + q^5 = 0,$$

- and *three* white combinations of colour with its anti-colour,

$$q + q^4 = 0, \; q^2 + q^5 = 0, \; q^3 + q^6 = 0,$$

just like a fermion and its antiparticle, or three bosons (like e.g. mesons π^0, π^+ and π^-).

Alternative proposal: colors first.

The generalized Dirac equation incorporating colour degrees of freedom in a Z_3-symmetric way was proposed in publications cited above; after introducing three pairs of independent Pauli spinors

$$\varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \quad \varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \quad \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix},$$

$$\chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}. \qquad (61)$$

with Pauli sigma-matrices acting on them in a natural way.

- **These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".**

- These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".
- The cyclic group Z_3 is represented on the complex plane by multiplicative group of three complex numbers, generated by powers of $j = e^{\frac{2\pi i}{3}}$, namely:

$$j = e^{\frac{2\pi i}{3}}, \;\; j^2 = e^{\frac{4\pi i}{3}}, \;\; j^3 = 1, \;\; 1 + j + j^2 = 0. \tag{62}$$

The resulting system of equation is as follows:

$$
\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\chi_- &= -j\;mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+,\\
E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\psi_- &= -j^2\;mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned}
\tag{63}
$$

The color content is better seen in the following alternative basis:

$$
\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\,mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\chi_- &= -j\,mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+\\
E\,\psi_+ &= j^2\,mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\psi_- &= -j^2\,mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned}
\tag{64}
$$

- The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-colour symmetry is realized by multiplication of mass m by j each time the colour changes, i.e. more explicitly, Z_3 symmetry is realized as follows:

- **The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-colour symmetry is realized by multiplication of mass m by j each time the colour changes, i.e. more explicitly, Z_3 symmetry is realized as follows:**
-

$$m \to jm, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm, \tag{65}$$

$$m \to j^2 m, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm, \tag{66}$$

- The energy operator is obviously diagonal, and its action on the spinor-valued column-vector can be represented as a 6×6 operator valued unit matrix. The mass operator is diagonal, too, but its elements represent all powers of the **sixth root of unity** $q = e^{\frac{2\pi i}{6}}$, which are

$$q = -j^2,\ q^2 = j,\ q^3 = -1,\ q^2 = j^2,\ q^5 = -j \text{ and } q^6 = 1.$$

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$$q = -j^2,\ q^2 = j,\ q^3 = -1,\ q^2 = j^2,\ q^5 = -j \text{ and } q^6 = 1.$$

- The system (63) was formulated in a basis in which the "coloured" Pauli spinors alternate with their antiparticles; however, if we want to put forward the colour content, it is better to choose an alternative basis in the space of spinors arranged as follows:

$$(\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-)^T . \tag{67}$$

Then the mass and momentum operators take on the following form:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & jm & 0 & 0 & 0 \\ 0 & 0 & 0 & -jm & 0 & 0 \\ 0 & 0 & 0 & 0 & j^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & -j^2 m \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- **The dimension of the two matrices M and P displayed above is 12×12: all the entries in the first one are proportional to the 2×2 identity matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\, m$, etc.**

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- **The entries in the second matrix P contain 2×2 Pauli's sigma-matrices, so that P is also a 12×12 matrix. The energy operator E is proportional to the 12×12 identity matrix.**

- Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of circulant 3×3 circulant matrix that are multiplicities of 3 are proportional to $\mathbb{1}_3$.
  The diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,

- Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of circulant 3×3 circulant matrix that are multiplicities of 3 are proportional to $\mathbb{1}_3$.
  The diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,
-

$$E^6 \, \varphi_+ = m^6 c^{12} \, \varphi_+ + c^6 \mid \mathbf{p} \mid^6 \, \varphi_+,$$
$$E^6 \, \varphi_- = m^6 c^{12} \, \varphi_- + c^6 \mid \mathbf{p} \mid^6 \, \varphi_-. \qquad (68)$$

and similarly for all other components.

Using a more rigorous approach the three operators can be expressed in terms of tensor products of matrices of lower dimensions. Let us introduce two following 3×3 matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{69}$$

whose products and powers generate the $U(3)$ Lie group algebra, or the $SU(3)$ algebra if we remove the unit matrix.

The standard 3×3 matrix basis of ternary Clifford algebra (first considered in XIX-th century by Cayley and Sylvester, who called its elements "*nonions*") looks as follows:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \; Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{70}$$

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \; Q_2^\dagger = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \; Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{71}$$

where j is the third primitive root of unity,

$$j = e^{\frac{2\pi i}{3}}, \;\; j^2 = e^{\frac{4\pi i}{3}}, \;\; 1 + j + j^2 = 0. \tag{72}$$

and $\mathcal{M}^\dagger$ denotes the Hermitean conjugate of matrix $\mathcal{M}$. We see that all the matrices Q and $Q^\dagger$ are non-Hermitean.

To complete the basis of 3×3 traceless matrices, we must add to Q and $Q^\dagger$ the following two linearly independent *diagonal* matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}. \tag{73}$$

We shall also use alternative notation $I_A,\ A = 1, 2, ...8$, with

$$I_1 = Q_1,\ I_2 = Q_2, I_3 = Q_3,\ I_4 = Q_1^\dagger,\ I_5 = Q_2^\dagger,\ I_6 = Q_6^\dagger,\ I_7 = B,\ I_8 = B^\dagger \tag{74}$$

and can also add $I_0 = \mathbb{1}_3$. The Hermitean conjugation $I_A^\dagger\ (A = 1, 2, ..., 8)$:

$$I_A^\dagger = (Q_1^\dagger,\ Q_2^\dagger,\ Q_3^\dagger,\ Q_1,\ Q_2,\ Q_3,\ B^\dagger,\ B) = I_{A^\dagger} \tag{75}$$

provides the following permutation of indices $A \rightarrow A^\dagger$:

$$A = (1, 2, 3, 4, 5, 6, 7, 8) \rightarrow A^\dagger = (4, 5, 6, 1, 2, 3, 8, 7). \tag{76}$$

We can introduce as well the standard complex conjugation $\mathcal{M} \rightarrow \bar{\mathcal{M}}$, which leads to the relations

$$\bar{I}_A = (\bar{Q}_1 = Q_2, \bar{Q}_2 = Q_1, \bar{Q}_3 = Q_3, \quad \bar{Q}_1^\dagger = Q_2^\dagger, \; \bar{Q}_2^\dagger = Q_1^\dagger \;\; \bar{B} = B^\dagger) = I_{\bar{A}}, \tag{77}$$

which corresponds to another permutation of indices A,

$$A = (1,2,3,4,5,6,7,8) \rightarrow \bar{A} = (2,1,3,5,4,6,8,7). \tag{78}$$

The 3×3 matrices Q_3 and $Q_3^\dagger$ are real, while $Q_2 = \bar{Q}_1$ are mutually complex conjugated, as well as their Hermitean counterparts $Q_2^\dagger = \bar{Q}_1^\dagger$.

The six matrices Q_k and $Q_j^\dagger$, $i, j = 1, 2, 3$ are endowed with natural $\mathbb{Z}_3$-grading

$$\text{grade}(Q_i) = 1, \quad \text{grade}(Q_k^\dagger) = 2, \tag{79}$$

Out of three independent $\mathbb{Z}_3$-grade 0 ternary (i.e. three-linear) combinations, only one leads to a non-vanishing result. One can simply check that both j and j^2 *ternary skew commutators* do vanish

$$\{Q_1, Q_2, Q_3\}_j = Q_1 Q_2 Q_3 + j Q_2 Q_3 Q_1 + j^2 Q_3 Q_1 Q_2 = 0, \tag{80}$$

$$\{Q_1, Q_2, Q_3\}_{j^2} = Q_1 Q_2 Q_3 + j^2 Q_2 Q_3 Q_1 + j Q_3 Q_1 Q_2 = 0, \tag{81}$$

as well as the odd permutation, e.g.
$Q_2 Q_1 Q_3 + j Q_1 Q_3 Q_2 + j^2 Q_3 Q_2 Q_1 = 0.$

In contrast, the totally symmetric combination does not vanish but is proportional to the 3×3 identity matrix $I_0 = \mathbb{1}_3$:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = 3\, \eta_{abc}\, \mathbb{1}_3, \quad a, b, ... = 1, 2, 3. \quad (82)$$

with η_{abc} given by the following non-zero components

$$\eta_{111} = \eta_{222} = \eta_{333} = 1, \;\; \eta_{123} = \eta_{231} = \eta_{312} = j^2,$$
$$\eta_{213} = \eta_{321} = \eta_{132} = j \quad (83)$$

and all other components vanishing. The above relation can be used as definition of *ternary Clifford algebra* .

An analogous set of relations is formed by Hermitean conjugates $Q^{\dagger}_{\dot{a}} := \bar{Q}^T_a$ of matrices Q_a, which we shall endow with dotted indices $\dot{a}, \dot{b}, ... = 1, 2, 3$. They satisfy the relation

$$Q_a^2 = Q^{\dagger}_{\dot{a}} \tag{84}$$

as well as the identities conjugate to the ones in (82)

$$Q^{\dagger}_{\dot{a}} Q^{\dagger}_{\dot{b}} Q^{\dagger}_{\dot{c}} + Q^{\dagger}_{\dot{b}} Q^{\dagger}_{\dot{c}} Q^{\dagger}_{\dot{a}} + Q^{\dagger}_{\dot{c}} Q^{\dagger}_{\dot{a}} Q^{\dagger}_{\dot{b}} = 3\, \eta_{\dot{a}\dot{b}\dot{c}}\, \mathbb{1}_3, \text{with } \ \eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{cba}. \tag{85}$$

The eight matrices $Q_a, \; Q^{\dagger}_b, \; B$ and $B^{\dagger}$ span a basis of the $SU(3)$ Lie algebra, with $B, \; B^{\dagger}$ its Cartan subalgebra. The commonly used basis of traceless Gell-Mann matrices $\lambda_{\alpha}, \;\; \alpha = 1, 2, ...8$ can be obtained by linear combinations of $Q_a, \; Q^{\dagger}_b, \; B, \; B^{\dagger}$

The full multiplication table of matrices $Q_a, \; Q^{\dagger}_b, \; B$ and $B^{\dagger}$ is given in the next slide.

$$
\begin{aligned}
\lambda_1 &= \frac{1}{3}\left(Q_1 + Q_2 + Q_3 + Q_1^\dagger + Q_2^\dagger + Q_3^\dagger\right), \\
\lambda_2 &= \frac{i}{3}\left(Q_1 + Q_2 + Q_3 - Q_1^\dagger - Q_2^\dagger - Q_3^\dagger\right), \\
\lambda_4 &= \frac{1}{3}\left(jQ_1 + j^2 Q_2 + Q_3 + j^2 Q_1^\dagger + jQ_2^\dagger + Q_3^\dagger\right), \\
\lambda_5 &= \frac{i}{3}\left(jQ_1 + j^2 Q_2 + Q_3 - j^2 Q_1^\dagger - jQ_2^\dagger - Q_3^\dagger\right), \\
\lambda_6 &= \frac{1}{3}\left(j^2 Q_1 + jQ_2 + Q_3 + jQ_1^\dagger + j^2 Q_2^\dagger + Q_3^\dagger\right), \\
\lambda_7 &= \frac{i}{3}\left(jQ_1 + j^2 Q_2 + Q_3 - j^2 Q_1^\dagger - jQ_2^\dagger - Q_3^\dagger\right),
\end{aligned}
\tag{86}
$$

	Q_1	Q_2	Q_3	$Q_1^\dagger$	$Q_2^\dagger$	$Q_3^\dagger$	B	$B^\dagger$
Q_1	$Q_1^\dagger$	$j^2\ Q_3^\dagger$	$j\ Q_2^\dagger$	$1\!\!1_3$	$B^\dagger$	B	$j\ Q_2$	$j^2\ Q_3$
Q_2	$j\ Q_3^\dagger$	$Q_2^\dagger$	$j^2\ Q_1^\dagger$	B	$1\!\!1_3$	$B^\dagger$	$j\ Q_3$	$j^2\ Q_1$
Q_3	$j^2 Q_2^\dagger$	$j\ Q_1^\dagger$	$Q_3^\dagger$	$B^\dagger$	B	$1\!\!1_3$	$j\ Q_1$	$j^2\ Q_2$
$Q_1^\dagger$	$1\!\!1_3$	$j^2\ B$	$j\ B^\dagger$	Q_1	$j^2\ Q_3$	$j\ Q_2$	$Q_3^\dagger$	$Q_2^\dagger$
$Q_2^\dagger$	$j\ B^\dagger$	$1\!\!1_3$	$j^2\ B$	$j\ Q_3$	Q_2	$j^2\ Q_1$	$Q_1^\dagger$	$Q_3^\dagger$
$Q_3^\dagger$	$j^2\ B$	$j\ B^\dagger$	$1\!\!1_3$	$j^2\ Q_2$	$j\ Q_1$	Q_3	$Q_2^\dagger$	$Q_1^\dagger$
B	Q_2	Q_3	Q_1	$j\ Q_3^\dagger$	$j\ Q_1^\dagger$	$j\ Q_2^\dagger$	$B^\dagger$	$1\!\!1_3$
$B^\dagger$	Q_3	Q_1	Q_2	$j^2\ Q_2^\dagger$	$j^2\ Q_3^\dagger$	$j^2\ Q_1^\dagger$	$1\!\!1_3$	B

As can be easily seen, the new matrices generated by multiplication of the 8 still do not form a group. Besides the initial set $Q_a, Q_a^\dagger, B$ and $B^\dagger$ their products yield the new matrices obtained by multiplication of the above by j and j^2, and the unit matrix $\mathbb{1}_3$. However, the matrices jB and $j^2 B^\dagger$ are missing, as well as the matrices $j \mathbb{1}_3$ and $j^2 \mathbb{1}_3$

They do appear after iteration, as products of some of the new matrices contained in the Table 2, e.g. $(j\,Q_1) \cdot (j\,Q_1^\dagger) = j^2 \mathbb{1}_3$, and so on. Notice that the multiplication by j or j^2 keeps the fundamental $SU(3)$ property valid: if $Q_a^\dagger = Q_a^{-1}$, similarly $(j\,Q_a)^\dagger = j^2\,Q_a^\dagger = (j\,Q_a)^{-1}$, and their determinants are also equal to 1, because $j^3 = 1$ and $(j^2)^3 = 1$ as well.

The full set of 27 elements forms a finite group, which is isomorphic with the tensor product of 9 nonion matrices by the Z_3 cyclic group. This group is obviously a finite subgroup of the fundamental representation of the Lie group $SU(3)$.

The 12×12 matrices M and P can be represented as the following tensor products:

$$M = m\, B \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad P = Q_3 \otimes \sigma_1 \otimes (\boldsymbol{\sigma} \cdot \mathbf{p}) \tag{87}$$

with as usual,

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Let us rewrite the matrix operator generating the system **(64)** when it acts on the column vector containing twelve components of three "colour" fields, in the basis **(67)** $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:

- Let us rewrite the matrix operator generating the system **(64)** when it acts on the column vector containing twelve components of three "colour" fields, in the basis **(67)** $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:
-

$$E \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 = mc^2 \; B \otimes \sigma_3 \otimes \mathbb{1}_2 + Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p}$$

with energy and momentum operators on the left hand side, and the mass operator on the right hand side:

$$E \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2 - Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \; B \otimes \sigma_3 \otimes \mathbb{1}_2 \quad (88)$$

- **Like with the standard Dirac equation, let us transform this equation so that the mass operator becomes proportional the the unit matrix. To do so, we multiply the equation (88) from the left by the matrix** $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.

- Like with the standard Dirac equation, let us transform this equation so that the mass operator becomes proportional the the unit matrix. To do so, we multiply the equation **(88)** from the left by the matrix $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.
- Now we get the following equation which enables us to interpret the energy and the momentum as the components of a Minkowskian four-vector $c\, p^\mu = [E,\ c\mathbf{p}]$:

$$E\ B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\,\boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \quad (89)$$

where we used the fact that under matrix multiplication, $\sigma_3\sigma^3 = \mathbb{1}_2,\ B^\dagger B = \mathbb{1}_3$ and $B^\dagger Q_3 = Q_2$.

- **The sixth power of this operator gives the same result as before,**

$$\left[E\, B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p}\right]^6 = \left[E^6 - c^6 \mathbf{p}^6\right] \mathbb{1}_{12}$$
$$= m^6 c^{12} \mathbb{1}_{12} \qquad (90)$$

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- It is also worth to note that taking the determinant on both sides of the eq. **(89)** yields the twelfth-order equation:

$$\left(E^6 - c^6\ \mid \mathbf{p} \mid^6\right)^2 = m^{12} c^{24}. \qquad (91)$$

- The equation **(89)** can be written in a concise manner using the Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

$$\Gamma^{\mu} p_{\mu} \, \Psi = mc \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \tag{92}$$

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- with 12×12 matrices $\Gamma^{\mu}, \;\; (\mu = 0, 1, 2, 3)$ defined as follows:

$$\Gamma^0 = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \tag{93}$$

There is still certain arbitrariness in the choice of 3×3 matrix factors $B^\dagger$ and Q_2 in the colour Dirac operator.

This is due to the choice of $j = e^{\frac{2\pi i}{3}}$ as the generator of the representation of the finite Z_3-symmetry group.

If j^2 is chosen instead, in **(89)** the matrix $B^\dagger$ will be replaced by B, Q_2 by Q_1, which is its complex conjugate; the remaining terms keep the same form.

The equation **(88)** can be written in a concise manner by introducing the 12×12 **matrix** colour Dirac operator $\Gamma^\mu p_\mu$ using Minkowskian indices and metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$:

$$\Gamma^\mu p_\mu \, \Psi = mc \, \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \qquad (94)$$

with 12×12 **matrices** Γ^μ $(\mu = 0, 1, 2, 3)$ defined as follows:

$$\Gamma^0 = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \qquad (95)$$

The 12-component colour Dirac equation is invariant under an arbitrary similarity transformation, i.e. if we set

$$\Psi' = \mathcal{R}\,\Psi, \quad (\Gamma^\mu)' = \mathcal{R}\,\Gamma^\mu\,\mathcal{R}^{-1} \quad \text{then} \quad (\Gamma^\mu)' p_\mu\,\Psi' = mc\,\Psi', \tag{96}$$

we get obviously

$$\left[(\Gamma^\mu)' p_\mu\right]^6 = (p_0^6 - \mid \mathbf{p} \mid^6) \mathbb{1}_{12} \tag{97}$$

Following the formulae (95) for the colour Dirac Γ^μ-matrices we see that they are neither real ($\bar{\Gamma}^\mu \neq \Gamma^\mu$) nor Hermitean ($(\Gamma^\mu)^\dagger \neq \Gamma^\mu$).

From the colour Dirac equation (89) one gets the equations for complex-conjugated $\bar{\Psi}$ and Hermitean-conjugated $\Psi^\dagger$:

$$\bar{\Gamma}^\mu p_\mu \, \bar{\Psi} = mc \, \bar{\Psi}, \qquad p_\mu \Psi^\dagger (\Gamma^\mu)^\dagger = mc \Psi^\dagger, \tag{98}$$

where $\bar{\Psi}$ **is a column** , $\Psi^\dagger$ **is a row** , $\bar{\sigma}_k = -\sigma_2 \sigma_k \sigma_2$, $\sigma_k = \sigma^k$, $\sigma_0 = \sigma^0 = \mathbb{1}_2$, **and**

$$\bar{\Gamma}^0 = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \bar{\Gamma}^k = Q_1 \otimes (i\sigma_2) \otimes \bar{\sigma}^k,$$

$$(\Gamma^0)^\dagger = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad (\Gamma^k)^\dagger = Q_1 \otimes \sigma_3 \otimes \sigma^k, \tag{99}$$

The second equation of **(98)** can be written in terms of matrices Γ^μ if we introduce the Hermitean-adjoint colour Dirac spinor $\Psi^H = \Psi^\dagger C$, where the 12×12-**matrix** C satisfies the relation

$$(\Gamma^\mu)^\dagger C = C \Gamma^\mu. \tag{100}$$

It can be also shown that neither $\bar{\Gamma}^\mu$ nor $(\Gamma^\mu)^\dagger$ can be obtained via similarity transformation.

To obtain a general solution of the colour Dirac equation one should use its Fourier transformed version. In the momentum space it becomes:

$$\left(\Gamma^{\mu}\, p_{\mu} - m \mathbb{1}_{12}\right)\, \hat{\Psi}(p) = 0. \tag{101}$$

The sixth power of the matrix $\Gamma^{\mu} p_{\mu}$ is diagonal and proportional to m^6, so that we have

$$(\Gamma^{\mu} p_{\mu})^6 - m^6 \mathbb{1}_{12} = \left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right) \mathbb{1}_{12} = 0. \tag{102}$$

Now we should find the inverse of the matrix $(\Gamma^\mu p_\mu - m \mathbb{1}_{12})$. Let us note that the sixth-order expression on the left-hand side in (102) can be factorized as follows:

$$(\Gamma^\mu p_\mu)^6 - m^6 = \left((\Gamma^\mu p_\mu)^2 - m^2\right) \left((\Gamma^\mu p_\mu)^2 - j\, m^2\right) \left((\Gamma^\mu p_\mu)^2 - j^2\, m^2\right). \tag{103}$$

The first factor can be expressed as the product of two linear operators, one of which defines the colour Dirac equation (92), (101):

$$(\Gamma^\mu p_\mu)^2 - m^2 = (\Gamma^\mu p_\mu - m)\ (\Gamma^\mu p_\mu + m) \qquad (104)$$

Therefore the inverse of the Fourier transform of the linear operator defining the colour Dirac equation **(101)** is given by the following matrix:

$$[\Gamma^\mu p_\mu - m]^{-1} = \frac{(\Gamma^\mu p_\mu + m)\ \left((\Gamma^\mu p_\mu)^2 - j\ m^2\right)\ \left((\Gamma^\mu p_\mu)^2 - j^2\ m^2\right)}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)}. \qquad (105)$$

- **The inverse of the six-order polynomial can be decomposed into a sum of three expressions with second-order denominators, multiplied by the common factor of the fourth order. Let us denote by Ω the sixth root of** $(| \mathbf{p} |^6 + m^6)$,

$$\Omega = \sqrt[6]{| \mathbf{p} |^6 + m^6}, \tag{106}$$

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$$\Omega = \sqrt[6]{\mid \mathbf{p} \mid^6 + m^6}, \tag{106}$$

- along with five other root values obtained via multiplication by consecutive powers of the sixth root of unity, $q = e^{\frac{2\pi i}{6}}$. Recalling the definition of j and that $q^2 = j$, we have the identity

$$(p_0^6 - \Omega^6) = (p_0^2 - \Omega^2)((p_0^2 - j\Omega^2)((p_0^2 - j^2\Omega^2) \tag{107}$$

which leads to the decomposition formula

$$\frac{1}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)} = \frac{1}{3\,\Omega^4}\left[\frac{1}{p_0^2 - \Omega^2} + \frac{j}{p_0^2 - j\Omega^2} + \frac{j^2}{p_0^2 - j^2\Omega^2}\right] \tag{108}$$

or equivalently,

$$\frac{1}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)} = \frac{1}{3\,\Omega^4}\left[\frac{1}{p_0^2 - \Omega^2} + \frac{1}{j^2 p_0^2 - \Omega^2} + \frac{1}{j p_0^2 - \Omega^2}\right] \tag{109}$$

After such a substitution in (105), six Z_6-graded simple poles do appear, Figure (4) illustrating the location of these six poles in the complex energy plane.

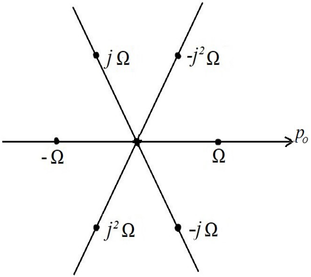

Figure: The six simple poles in the Fourier-transform of the propagator (108), with two real ones $\pm\Omega$ and two conjugate Lee-Wick poles $\pm j\Omega$, $\pm j^2\Omega$.

As long as there is a non-zero mass term, we do not encounter the infrared divergence problem at $\mid \mathbf{p} \mid \to 0$. Each of the three inverses of a second-order polynomial can be in turn expressed as a sum of simple first-order poles, e.g.

$$\frac{1}{p_0^2 - j\Omega^2} = \frac{j}{2\,\Omega}\left[\frac{1}{p_0 - j^2\Omega} - \frac{1}{p_0 + j^2\Omega}\right] = \frac{j^2}{2\,\Omega}\left[\frac{1}{jp_0 - \Omega} - \frac{1}{jp_0 + \Omega}\right], \tag{110}$$

and similarly for other terms in (108).

- In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane. The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 component (the energy) into the complex domain.

- **In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane. The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 component (the energy) into the complex domain.**
- **The first term in the decomposition (108) of the coulour Dirac propagator presents two simple poles on the real line, while the second and the third terms display two simple poles each, located on complex straight lines** $Imp_0 = jRep_0$ **and** $Imp_0 = j^2 Rep_0$.

- One can add that in the propagators given by formula **(108)** the non-standard residua $\pm j$ and $\pm j^2$ should be justified by suitable form of the Z_3-**graded commutators** describing quantum oscillator algebra of colour quark field excitations.

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- The colour Dirac equation (98) breaks the Lorentz symmetry $\mathcal{O}(1,3)$ reducing it to $\mathcal{O}_3$, because the 3×3-matrices describing "colour" are *different* for the Γ^0 and Γ^k components.
- However we shall show in the following Section that one can introduce a Z_3-graded generalization of the Lorentz transformations, acting in covariant way on three "replicas" of the energy-momentum four-vector introduced above.

The mass shell condition

$$E^6 - c^6 \mid \mathbf{p} \mid^6 = m^6 c^{12} \tag{111}$$

can be decomposed into the usual relativistic Klein-Gordon invariant multiplied by a strictly positive factor:

$$C_6 = p_0^6 - \Omega^6 = (p_0^2 - \mid \mathbf{p} \mid^2)(p_0^4 + p_0^2 \mid \mathbf{p} \mid^2 + \mid \mathbf{p} \mid^4) = m^6 c^6, \tag{112}$$

The sixth-order polynomial C_6 can be further decomposed into the product of the following three second-order polynomials,

$$C_6 = \overset{(0)}{C_2}\ \overset{(1)}{C_2}\ \overset{(2)}{C_2}\ , \tag{113}$$

$$\text{with} \quad \overset{(0)}{C_2} = p_0^2 - \mathbf{p}^2, \quad \overset{(1)}{C_2} = j\, p_0^2 - \mathbf{p}^2, \quad \overset{(2)}{C_2} = j^2\, p_0^2 - \mathbf{p}^2. \tag{114}$$

Let us denote by superscripts (0), (1) **and** (2) **the four-momenta with quadratic invariants given by** $\overset{(0)}{C_2}$, $\overset{(1)}{C_2}$ and $\overset{(2)}{C_2}$. **We get explicitly**

$$(p_0)^2 - (\overset{(0)}{\mathbf{p}})^2 = \overset{(0)}{C_2} \; ,$$

$$(\overset{(1)}{p_0})^2 - (\overset{(1)}{\mathbf{p}})^2 = \overset{(1)}{C_2} \; ,$$

$$(\overset{(2)}{p_0})^2 - (\overset{(2)}{\mathbf{p}})^2 = \overset{(2)}{C_2} \; , \qquad (115)$$

From any real four-vector $\overset{(0)}{p}_{0\,\mu}$ one can define its two **"replicas"**
$0\,\overset{(1)}{\mathbf{p}}_{\mu}$ and $\overset{(2)}{\mathbf{p}}_{\mu}$ with p_0 in the complex plane, obtained by rotations by j and by j^2 as follows:
Let us introduce three 4×4 **matrices** acting on Minkowskian four-vectors:

$$\overset{(0)}{A} = \mathrm{diag}\,(1,1,1,1) = 1\!\!1_4, \quad \overset{(1)}{A} = \mathrm{diag}\,(j^2,1,1,1), \quad \overset{(2)}{A} = \mathrm{diag}\,(j,1,1,1), \tag{116}$$

providing a (reducible) matrix representation of the cyclic Z_3 group,

$$\overset{(r)}{A}\,\overset{(s)}{A} = \overset{(r+s)}{A}\,. \tag{117}$$

The superscripts $(r+s)$ are added modulo 3, e.g.
$1+2 \to 0,\; 2+2 \to 1$, etc.

Acting on a given four-vector $p_\mu = (p_0, \mathbf{p})$ by one of the matrices $\overset{(r)}{A}$ we produce its three Z_3-graded "replicas" belonging correspondingly to sectors $\overset{(r)}{C_2}$:

$$\overset{(r)}{p} = \overset{(r)}{A} p : \quad \overset{(0)}{p_\mu} \to \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}, \quad \overset{(1)}{p_\mu} \to \begin{pmatrix} j^2 p_0 \\ \mathbf{p} \end{pmatrix}, \quad \overset{(2)}{p_\mu} \to \begin{pmatrix} j p_0 \\ \mathbf{p} \end{pmatrix}. \tag{118}$$

In what follows, we shall use a short-hand notation:

$$p'_\mu = L_\mu{}^\nu p_\nu \to p' = Lp, \qquad \overset{(r)}{p'}_\mu = \overset{(r)}{A}_\mu{}^\nu \overset{(0)}{p_\nu} \to \overset{(r)}{p'} = \overset{(r)}{A} \overset{(0)}{p} \tag{119}$$

It should be stressed here that the spacetime remains Minkowskian, with one real time and three real spatial coordinates; however, the components of $\overset{(1)}{p}_\mu$ and $\overset{(2)}{p}_\mu$ can take on particular Z_3-graded complex values.

Three "replicas" **(118)** are the images of the same four-vector which can be obtained by Z_3-valued rotations in the complex energy plane.

Let us denote by $\overset{(0)}{L}_{00}$ the classical Lorentz transformations which map the real Minkowskian momenta $\overset{(0)}{p}_\nu$ into $\overset{(0)}{p}{}'_\nu$

$$(\overset{(0)}{L}_{00})_\mu{}^\nu \overset{(0)}{p}_\nu = \overset{(0)}{p}{}'_\mu \rightarrow \overset{(0)}{L}_{00} \overset{(0)}{p} = \overset{(0)}{p}{}' , \qquad (120)$$

where lower indices (00) mean that we transform $\overset{(0)}{C}_2$ into itself, and the superscript (0) says that we deal with the classical Lorentz transformations.

The zero-grade Lorentz transformations can be extended to the mappings of four-vectors $\overset{(r)}{p}$ belonging to sector $\overset{(r)}{C_2}$ onto four-vectors $\overset{(s)}{p'}$ belonging to sector $\overset{(s)}{C_2}$, with $r, s = 0, 1, 2$. **Let us apply a Lorentz boost transforming a four-vector from the sector s, $\overset{(s)}{p}$ into a vector from another sector r, $\overset{(r)}{p'}$. With notations notations using the definition of A-matrices , we have:**

$$\overset{(0)}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overset{(1)}{A} = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overset{(2)}{A} = \begin{pmatrix} j^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the following formula

$$\overset{(r)}{p'} = \overset{(r)}{A}\,\overset{(0)}{p} = \overset{(r)}{A}\,\overset{(0)}{L_{00}}\,\overset{(0)}{p} = \overset{(r)}{A}\,\overset{(0)}{L_{00}}\,\overset{(s)}{A^{-1}}\,\overset{(s)}{A}\,\overset{(0)}{p} = \overset{(r-s)}{L_{rs}}\,\overset{(s)}{p} \tag{121}$$

$$\text{where} \quad \overset{(r-s)}{L_{rs}} = \overset{(r)}{A}\,\overset{(0)}{L_{00}}\,\overset{(s)}{A^{-1}}, \tag{122}$$

describes the Lorentz transformation from sector s onto sector r, and where the superscript $(r-s)$ accordingly to the Z_3-grading is taken modulo 3.

- **To provide the formulae for Z_3-graded boosts in explicit form we choose the four-vectore $p_\mu = (p_0, \mathbf{p})$ restricted to the plane $(0,1)$, with the three-vector $\mathbf{p}$ aligned along the first spatial axis.**

- To provide the formulae for Z_3-**graded boosts** in explicit form we choose the four-vectore $p_\mu = (p_0, \mathbf{p})$ restricted to the plane $(0,1)$, with the three-vector $\mathbf{p}$ aligned along the first spatial axis.
- In such a frame the Lorentz rotations reduce only to the boost in $(0,1)$ **plane**, given by the following transformation:

$$\begin{pmatrix} p'_0 \\ p'_1 \end{pmatrix} = \begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}, \tag{123}$$

Subsequently, we get the following triplet of homogeneous transformations:

$\overset{(0)}{L}_{00}$, $\overset{(0)}{L}_{11}$ and $\overset{(0)}{L}_{22}$:

$$\overset{(0)}{L}_{00}(u) = \begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(0)}{L}_{11}(u) = \begin{pmatrix} \mathrm{ch}u & j^2\mathrm{sh}u \\ j\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(0)}{L}_{22}(u) = \begin{pmatrix} \mathrm{ch}u & j\mathrm{sh}u \\ j^2\mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \tag{124}$$

preserving respectively the bilinear forms $\overset{(r)}{C}_2$.

- The matrices **(124)** are self-adjoint:

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}, \quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}, \quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22} \qquad (125)$$

- The matrices **(124)** are self-adjoint:

$$\overset{(0)\dagger}{L_{00}} = \overset{(0)}{L_{00}}, \quad \overset{(0)\dagger}{L_{11}} = \overset{(0)}{L_{11}}, \quad \overset{(0)\dagger}{L_{22}} = \overset{(0)}{L_{22}} \qquad (125)$$

- The generalized Lorentz boosts **(124)** conserve the group property: the product of two Lorentz boosts acting in the r-th sector is a boost of the same type. Indeed, we see from **(124)** that the product of two boosts acting in the r-th sector **(**$r = 0, 1, 2$**)** looks as follows (no summation over r!):

$$\overset{(0)}{L_{rr}}(u) \cdot \overset{(0)}{L_{rr}}(v) = \overset{(0)}{L_{rr}}(u+v). \qquad (126)$$

If we look at three fourdimensional Lorentz boost transformations on planes $(0, i)$, $i = 1, 2, 3$, the respective set of three independent "classical" Lorentz boosts belonging to $L_{00}^{(0)}$ requires the introduction of three 4×4 **matrices** with three independent parameters u, v, w:

$$\begin{pmatrix} \text{ch}u & \text{sh}u & 0 & 0 \\ \text{sh}u & \text{ch}u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \text{ch}v & 0 & \text{sh}v & 0 \\ 0 & 1 & 0 & 0 \\ \text{sh}v & 0 & \text{ch}v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \text{ch}w & 0 & 0 & \text{sh}w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \text{sh}w & 0 & 0 & \text{ch}w \end{pmatrix} \tag{127}$$

Next, let us consider the general set of matrices transforming the s-th sector into the r-th one,

$$\overset{(r)}{p'}_{\mu} = (\overset{(r-s)}{L_{rs}})_{\mu}{}^{\nu} \, \overset{(s)}{p}_{\nu} \, , \quad r, s = 0, 1, 2, \quad r \neq s. \tag{128}$$

There are two types of such matrices: raising and lowering the Z_3-grade by 1. For the sake of simplicity, let us firstly consider the two-dimensional case (i.e. $\mu, \nu = 0, 1$ in (128).

The 2×2-**matrices** raising the Z_3 **index** (r) of the generalized four-momenta $(\overset{(r)}{p}_\mu \rightarrow \overset{(r+1)}{p}_\mu$ are:

$$\overset{(1)}{L}_{10} = \begin{pmatrix} j^2 \mathrm{ch} u & j^2 \mathrm{sh} u \\ \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \quad \overset{(1)}{L}_{21} = \begin{pmatrix} j^2 \mathrm{ch} u & j \mathrm{sh} u \\ j \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \quad \overset{(1)}{L}_{02} = \begin{pmatrix} j^2 \mathrm{ch} u & \mathrm{sh} u \\ j^2 \mathrm{sh} u & \mathrm{ch} u \end{pmatrix} \tag{129}$$

The determinants of the matrices **(129)** are equal to j^2.

The matrices lowering the Z_3 **index** by one (or increasing it by 2, what is equivalent from the point of view of the Z_3-**grading**) are:

$$\overset{(2)}{L}_{01} = \begin{pmatrix} j\,\mathrm{ch}\,u & \mathrm{sh}\,u \\ j\,\mathrm{sh}\,u & \mathrm{ch}\,u \end{pmatrix}, \quad \overset{(2)}{L}_{12} = \begin{pmatrix} j\,\mathrm{ch}\,u & j^2\mathrm{sh}\,u \\ j^2\mathrm{sh}\,u & \mathrm{ch}\,u \end{pmatrix}, \quad \overset{(2)}{L}_{20} = \begin{pmatrix} j\,\mathrm{ch}\,u & j\,\mathrm{sh}\,u \\ \mathrm{sh}\,u & \mathrm{ch}\,u \end{pmatrix} \tag{130}$$

The determinants of the matrices (130) are equal to j.

The above two sets of three matrices each are mutually **Hermitean-adjoint**:

$$\overset{(1)\dagger}{L}_{01} = \overset{(2)}{L}_{10}\,, \quad \overset{(1)\dagger}{L}_{12} = \overset{(2)}{L}_{21}\,, \quad \overset{(1)\dagger}{L}_{20} = \overset{(2)}{L}_{02} \tag{131}$$

We recall that the superscript over each matrix $\overset{(t)}{L}_{rs}$ is equal to the difference of its lower indices, i.e. $(t) = (r - s)$.

The matrices $\overset{(1)}{L}_{rs}$ and $\overset{(2)}{L}_{rs}$ $(r, s = 0, 1, 2)$ raising or lowering respectively the Z_3-**grade** of the four-momentum vectors $\overset{(r)}{p}_{\mu}$ do not form a Lie group.

However, together with matrices $\overset{(0)}{L}_{rs}$ they can be used as building blocks in bigger 12×12 **matrices** forming a Z_3-**graded generalization** of the Lorentz group.

This construction is possible due to the chain rule obeyed by these matrices, which due to the definition (122) display the group property. We have:

$$\overset{(r-s)}{L_{rs}}\ (p_0, p_1;\ u)\ \overset{(s-t)}{L_{st}}\ (p_0, p_1;\ v) = \overset{(r-t)}{L_{rt}}\ (p_0, p_1;\ (u+v)). \quad (132)$$

In order to pass to arbitrary four-momentum vectors $\overset{(r)}{p_\mu}$, $\mu = 0, 1, 2, 3$ **one should embed the** 2×2 **matrices (129 - 130) into** 4×4 **matrices in a way analogous to passing from the** 2×2 **boost matrices** $\overset{(0)}{L_{00}}$ **to the triplet of boosts in planes** $(0, i), i = 1, 2, 3$ **described by the** 4×4 **matrices** (127).

If we write a Z_3-extended four-momentum vector $(\overset{(0)}{p^\mu}, \overset{(1)}{p^\mu}, \overset{(2)}{p^\mu})^T$ as a column with 12 entries, we can introduce three boost sectors $\overset{(r)}{\Lambda}$, $(r = 0, 1, 2)$ of the generalized Z_3-graded Lorentz group as 12×12 matrices as follows:

$$\overset{(0)}{\Lambda} : \begin{pmatrix} \overset{(0)}{L}_{00} & 0 & 0 \\ 0 & \overset{(0)}{L}_{11} & 0 \\ 0 & 0 & \overset{(0)}{L}_{22} \end{pmatrix} \overset{(1)}{\Lambda} : \begin{pmatrix} 0 & 0 & \overset{(1)}{L}_{02} \\ \overset{(1)}{L}_{10} & 0 & 0 \\ 0 & \overset{(1)}{L}_{21} & 0 \end{pmatrix} \overset{(2)}{\Lambda} : \begin{pmatrix} 0 & \overset{(2)}{L}_{01} & 0 \\ 0 & 0 & \overset{(2)}{L}_{12} \\ \overset{(2)}{L}_{20} & 0 & 0 \end{pmatrix}. \tag{133}$$

In each of the 12×12 matrices $\overset{(r)}{\Lambda}$, $r = 0, 1, 2$ the triplets of 4×4 matrices $\overset{(r-s)}{L_{rs}}$ are obtained from the standard classical Lorentz boosts by using the definition **(122)**, i.e. each $\overset{(r)}{\Lambda}$-matrix depends exclusively on three parameters defining three independent classical Lorentz boosts.

One can show that our matrices display the following Z_3-graded multiplication rules:

$$\overset{(0)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r)}{\boldsymbol{\Lambda}}, \quad \overset{(1)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+1)}{\boldsymbol{\Lambda}}, \quad \overset{(2)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+2)}{\boldsymbol{\Lambda}}, \tag{134}$$

where $\overset{(r)}{\boldsymbol{\Lambda}}$ $(r = 0, 1, 2)$ denote the Z_3-graded sectors of the full set of 12×12 matrix Lorentz group which includes also the Z_3-graded $O(3)$ spatial rotations.

The multiplication rules **(eq. 134)** with the Z_3-graded structure can be described in a compact way using the bold-face symbols $\overset{(r)}{\boldsymbol{\Lambda}}$ as follows:

$$\overset{(r)}{\boldsymbol{\Lambda}} \cdot \overset{(s)}{\boldsymbol{\Lambda}} \subset \overset{(r+s)|_3}{\boldsymbol{\Lambda}}, \quad \text{with} \;\; r, s, .. = 0, 1, 2, \;\; (r+s) \text{ taken modulo } 3. \tag{135}$$

The construction of Z_3-graded $O(3)$ rotations completing the Z_3-graded boosts $\overset{(r)}{\boldsymbol{\Lambda}}$ is a follows.

Let us denote by R_i the usual space rotation around the i-th axis, represented as a 3×3 matrix. When incorporated into the four-vector representation of the Lorentz group, it becomes a sub-matrix of a 4×4 Lorentzian matrix according to the formula $\overset{(0)}{R_i} = \begin{pmatrix} 1 & 0 \\ 0 & R_i \end{pmatrix}$. The Z_3-graded space rotations supplementing the Z_3-graded boosts **(133)** are constructed as the following 12×12 matrices:

$$\overset{(0)}{\mathcal{R}_i} = \mathbb{1}_3 \otimes \overset{(0)}{R_i}, \quad \overset{(1)}{\mathcal{R}_i} = Q_3^\dagger \otimes \overset{(0)}{R_i}, \quad \overset{(2)}{\mathcal{R}_i} = Q_3 \otimes \overset{(0)}{R_i}, \qquad (136)$$

where the choice of the colour generators $Q_3^\dagger$ and Q_3 is consistent with the initial definition of the colour Dirac equations.

The Z_3-**graded** infinitesimal generators of the Lorentz boosts can be obtained by considering the matrices $\overset{(r)}{\Lambda}$ with infinitesimal boost parameters what amounts to the replacements of the entries $\mathrm{sh}u$ by $\mathbf{1}$, and of all other entries, $\mathrm{ch}u$ and 1 alike, by 0, i.e. taking the differential. The resulting 12×12 **matrices** are the Lie algebra generators of the generalized Lorentz boosts, which we shall denote as $\overset{(r)}{K}_i$, $r = 0, 1, 2$. By taking their commutators we obtain the Z_3-**graded generators** of space rotations $(r + s)$ modulo 3):

$$[\overset{(r)}{K}_i, \overset{(s)}{K}_j] = -\epsilon_{ijk} \overset{(r+s)}{J}_k \tag{137}$$

In this way we obtained the full set of generators of the Z_3-graded Lorentz algebra which satisfy the following commutation relations:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)},$$
$$\left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{138}$$

which were firstly introduced and studied in: R. Kerner and J. Lukierski, Physics Letters B (2019)

Let us consider 12×12 component matrices $\overset{(r)}{\Lambda}$ as 3×3 matrices with their matrix entries represented by 4×4 blocks $\overset{(t)}{L}_{rs}$ (see (128))

The matrices $\overset{(0)}{\Lambda}$ are Hermitean by virtue of formula (125), while $(\overset{(1)}{\Lambda})^\dagger = \overset{(2)}{\Lambda}$ or equivalently, $(\overset{(2)}{\Lambda})^\dagger = \overset{(1)}{\Lambda}$ (see formula 131).

The group structure of 12×12 matrices $\mathbf{\Lambda} = (\overset{(0)}{\Lambda}, \overset{(1)}{\Lambda}, \overset{(2)}{\Lambda})$ is preserved under the similarity transformations,

$$\Lambda \rightarrow \tilde{\Lambda} = \mathcal{U} \Lambda \mathcal{U}^{-1}, \tag{139}$$

but the above Hermitean properties of Λ-matrices are conserved only if the transformation matrices are unitary. The proof is immediate: let us denote by $\mathcal{U} = U \otimes \mathbb{1}_4$ a 12×12 matrix where U is a 3×3 complex valued matrix by the unit 4×4 matrix $\mathbb{1}_4$ and denote $\mathcal{U}^\dagger = U^\dagger \otimes \mathbb{1}_4$.

Consider $\overset{(0)}{\Lambda} \to \mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$ and impose the **Hermiticity conditions** on the transformed matrices $\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$. The matrix $\overset{(0)}{\Lambda}$ being Hermitean, we get

$$\left(\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1} \right)^{\dagger} = (\mathcal{U}^{-1})^{\dagger} \overset{(0)}{\Lambda}{}^{\dagger} \mathcal{U}^{\dagger} = \mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}. \tag{140}$$

The matrix $\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$ is Hermitean as well if the similarity matrices $\mathcal{U}$ are *unitary*, i.e. if $\mathcal{U}^{\dagger} = \mathcal{U}^{-1}$, according to the formula $\mathcal{U} = U \otimes \mathbb{1}_4$ it follows that $U^{\dagger} = U^{-1}$. If the similarity matrices are unitary, the Hermitean conjugation relations between the matrices $\overset{(1)}{\Lambda}$ and $\overset{(2)}{\Lambda}$ are also preserved.

In this way we introduced the symmetry $SU(3)$ acting on the vector representation of the Z_3-graded Lorentz group. The 3×3 matrices U appearing in the 12×12 matrices $\mathcal{U}$ during the unitary similarity transformations leave the 4×4 Lorentzian blocks unaffected, in agreement with the well known "no-go theorems" by Coleman and Mandula and O'Raifeartaigh.

- **In order to obtain the entire Z_3-graded Lorentz group we should add as well the Z_3-graded extension of space rotations, also represented as 12×12 matrices, given by 3×3 matrices with 4×4-dimensional entries, as the Z_3-graded boosts.**

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- **As in the case of Lorentz boosts, besides the rotations that leave the transformed 3-momentum in the same sector, one gets also 12×12 matrices with non diagonal 4×4 entries (136), which map one of the Z_3-graded sector onto another one.**
- **We conclude that the full set of Z_3-graded $O(3)$ subgroup elements can be represented by 12×12 matrices and incorporated in the Z_3-graded Lorentz group.**

The Z_3-graded Lorentz group should also act on the coloured Dirac spinors through certain linear action, which should be realized as a generalized spinorial representatopn..

Such construction was introduced in our most recent publication:

R. Kerner and J. Lukierski, *Internal quark symmetries and colour* $SU(3)$ *entangled with* Z_3*-graded orentz algebra*, *Nuclear Physics B*, Vol. 972, (November 2021), 115529

which we shall briefly present in what follows.

What we want to find is a set of 12×12 **matrices** $J_i^{(r)}$, and $K_j^{(r)}$, $i, j.. = 0, 1, 2$ satisfying the same **Z_3-graded commutation relations** as their vectorial counterparts:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)}, \quad \left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{141}$$

The spinor representation of the zeroth sector $L^{(0)}$ of the Z_3-**graded Lorentz algebra** is obtained in a simplest possible manner, by tensorising the spinorial generators of the usual representation on Dirac spinors by the unit 3×3 **matrix**:

$$J_l^{(0)} = -\frac{i}{2} \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(0)} = -\frac{1}{2} \mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_i. \tag{142}$$

satisfying classical Lorentz algebra commutation relations:

$$\left[J_i^{(0)}, J_k^{(0)}\right] = \epsilon_{ikl} J_l^{(0)}, \quad \left[J_i^{(0)}, K_k^{(0)}\right] = \epsilon_{ikl} K_l^{(0)}, \\ \left[K_i^{(0)}, K_k^{(0)}\right] = -\epsilon_{ikl} J_l^{(0)}. \tag{143}$$

The two extra Lorentz sectors, $L^{(1)}$ and $L^{(2)}$, are constructed as the following 12×12 matrices:

$$J_l^{(1)} = -\frac{i}{2}\, Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(1)} = -\frac{1}{2}\, Q_3 \otimes \sigma_1 \otimes \sigma_i. \qquad (144)$$

$$J_i^{(2)} = -\frac{i}{2}\, Q_3^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_m^{(2)} = -\frac{1}{2}\, Q_3^\dagger \otimes \sigma_1 \otimes \sigma_m. \qquad (145)$$

Let us recall once more the notation $I_A, \; A = 1, 2, ...8$, with

$$I_1 = Q_1, \; I_2 = Q_2, I_3 = Q_3, \; I_4 = Q_1^\dagger, \; I_5 = Q_2^\dagger, \; I_6 = Q_6^\dagger, \; I_7 = B, \; I_8 = B^\dagger \tag{146}$$

We can also add $I_0 = \mathbb{1}_3$. The Hermitean conjugation $I_A^\dagger \;\; (A = 1, 2, ..., 8)$:

$$I_A^\dagger = (Q_1^\dagger, \; Q_2^\dagger, \; Q_3^\dagger, \; Q_1, \; Q_2, \; Q_3, \; B^\dagger, \; B) = I_{A^\dagger} \tag{147}$$

which provides the following permutation of indices $A \rightarrow A^\dagger$:

$$A = (1, 2, 3, 4, 5, 6, 7, 8) \rightarrow A^\dagger = (4, 5, 6, 1, 2, 3, 8, 7). \tag{148}$$

The formulae (143), (144) and (145) describe the spinorial realization of the Lie algebra $\mathcal{L}$ which is implied by the initial choice of matrices Γ^μ. Let us introduce a unified notation englobing all possible choices of Γ^μ-matrices $(A \neq B)$

$$\Gamma^0_{(A;\alpha)} = l_A \otimes \sigma_\alpha \otimes \sigma^0, \quad \Gamma^i_{(B;\beta)} = l_B \otimes (i\sigma_\beta) \otimes \sigma^i, \qquad (149)$$

where $l_0 = 1\!\!1_3$, l_A **with** $A = 1, 2, ..., 8$ colormagenta **are given in** (74), **and** $\alpha, \beta = 2, 3$ **but** $\{\sigma_\alpha,\ \sigma_\beta\}_+ = 0$ **i.e. we always have either** $\alpha = 2,\ \beta = 3$ **or** $\alpha = 3, \beta = 2$.

The choice $\alpha = 1$ is not present in the formula (149) because it is reserved for the description of symmetry generators $\mathcal{L}$ ((144), (145)). Further, eight colour 3×3 matrices l_A ($A = 1, 2, ...8$) span the ternary basis of the $SU(3)$ algebra.

- **The characteristic feature of "colour" Γ-matrices is that the 3×3 matrices I_A appearing as the first tensorial factors in (149) are *different* for temporal and spatial components of the matrix-valued 4-vector Γ^μ. We see that the choice of the colour factor in (149) depends on two sets of values of the four-vector index: $\mu = 0$ or $\mu = i, \;\; i = 1, 2, 3$. This property can be interpreted as the entanglement of colour and Lorentz symmetry degrees of freedom.**

- **The characteristic feature of "colour" Γ-matrices is that the 3×3 matrices I_A appearing as the first tensorial factors in (149) are *different* for temporal and spatial components of the matrix-valued 4-vector Γ^μ. We see that the choice of the colour factor in (149) depends on two sets of values of the four-vector index: $\mu = 0$ or $\mu = i, \quad i = 1, 2, 3$. This property can be interpreted as the entanglement of colour and Lorentz symmetry degrees of freedom.**
- **In the notation (150) basic Γ-matrices appearing in the first version of the colour Dirac equation can be denoted as**

$$\Gamma^0_{(8,3)} = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \Gamma^i_{(2;2)} = Q_2 \otimes (i\sigma_2) \otimes \sigma^i. \qquad (150)$$

- **In order to get a closed formula for the adjoint action $\mathcal{S}^{(0)}\Gamma^{\mu}[\mathcal{S}^{(0)}]^{-1}$ of classical spinorial Lorentz group, where $a^i, b^k, \ \ (i,k=1,2,3)$ are the six real $SL(2,\mathbf{C})$ Lie group parameters**

$$S^{(0)} = \exp\,\left(a^i K_i^{(0)} + b^k J_k^{(0)}\right) \qquad (151)$$

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$$S^{(0)} = \exp\left(a^i K_i^{(0)} + b^k J_k^{(0)}\right) \tag{151}$$

- we should introduce the following pairs of Γ^{μ}-matrices

$$\Gamma^{\mu} = (\Gamma^i_{(A;2)},\; \Gamma^0_{(B;3)}) \text{ and } \tilde{\Gamma}^{\mu} = (\Gamma^i_{(B;2)},\; \Gamma^0_{(A;3)}), \tag{152}$$

where we have chosen $\alpha = 3$ and $\beta = 2$.

Although for *any* choice of the first factor I_A in $\Gamma^{\mu}_{(A;\alpha)}$'s we have

$$\left[J_i^{(0)}, \Gamma^j_{(A;\alpha)} \right] = \epsilon_{ijk} \Gamma^k_{(A;\alpha)}, \quad \left[J_i^{(0)}, \Gamma^0_{(A;\alpha)} \right] = 0, \qquad (153)$$

the boosts $K_i^{(0)}$ act covariantly only on doublets $\left(\Gamma^{\mu}, \tilde{\Gamma}^{\mu} \right)$, with $(A \neq B)$, because only for such a choice we can get the closure of commutation relations:

$$[K_i^{(0)}, \Gamma^j_{(A;2)}] = \delta^j_i \, \Gamma^0_{(A;3)}, \quad [K_i^{(0)}, \Gamma^0_{(B;3)}] = \Gamma^i_{(B;2)},$$

$$[K_i^{(0)}, \Gamma^j_{(B;2)}] = \delta^j_i \, \Gamma^0_{(B;3)}, \; [K_i^{(0)}, \Gamma^0_{(A;3)}] = \Gamma^i_{(A;2)}. \qquad (154)$$

It follows from **(153), (154)** that the standard Lorentz covariance requires the pair of coloured Dirac equations described by the *doublet* $(\Gamma^\mu, \tilde{\Gamma}^\mu)$ of coloured Dirac matrices, which we shall call "*Lorentz doublets*". In particular, the Γ^μ matrices should be supplemented by the following Lorentz doublet partner:

$$\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^i_{(8;2)} = B^\dagger \otimes (i\sigma_2) \otimes \sigma^i. \quad (155)$$

- **The Lorentz doublets of Γ^{μ}-matrices required by the standard Lorentz covariance can be used for the description of weak isospin (flavour) doublets of the $SU(2) \times U(1)$ electroweak symmetry.**

- **The Lorentz doublets of Γ^{μ}-matrices required by the standard Lorentz covariance can be used for the description of weak isospin (flavour) doublets of the $SU(2) \times U(1)$ electroweak symmetry.**
- **In such a way one can show that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of Standard Model are linked with the presence of standard Lorentz covariance which generates three 24-component Lorentz doublets of colour Dirac spinors.**

- By calculating the multicommutators of $\left(J_i^{(1)}, K_l^{(1)}\right) \in L^{(1)}$ with the set $\Gamma^{\mu}_{(a)}$, $(a = 1, 2 ... 6)$, we will show that the following **sextet of Γ-matrices** which break the Lorentz covariance is closed under the action of $L^{(1)}$:

- By calculating the multicommutators of $\left(J_i^{(1)}, K_l^{(1)}\right) \in L^{(1)}$ with the set $\Gamma^{\mu}_{(a)}$, $(a = 1, 2 \ldots 6)$, we will show that the following sextet of Γ-matrices which break the Lorentz covariance is closed under the action of $L^{(1)}$:
-

$$\Gamma^{\mu}_{(1)} = \left(\Gamma^0_{(8;3)},\ \Gamma^i_{(2;2)}\right); \quad \Gamma^{\mu}_{(4)} = \left(\Gamma^0_{(8;2)},\ \Gamma^i_{(2;3)}\right);$$
$$\Gamma^{\mu}_{(2)} = \left(\Gamma^0_{(2;2)},\ \Gamma^i_{(4;3)}\right); \quad \Gamma^{\mu}_{(5)} = \left(\Gamma^0_{(2;3)},\ \Gamma^i_{(4;2)}\right); \tag{156}$$
$$\Gamma^{\mu}_{(3)} = \left(\Gamma^0_{(4;3)},\ \Gamma^i_{(8;2)}\right); \quad \Gamma^{\mu}_{(6)} = \left(\Gamma^0_{(4;2)},\ \Gamma^i_{(8;3)}\right).$$

It is easy to see that from the six components of the sextet **(156)** one can construct as well the set of six Γ^μ-matrices $\Gamma^\mu_{(A;\alpha)}$, $A = 2, 4, 8$ and $\alpha = 2, 3$, which can be described as well as three Lorentz doublets **(152)** , with $(A, B) = (2, 8), (2, 4)$ **and** $(4, 8)$**..** More explicitly,

$$\left(\Gamma^0_{(A;\alpha)} = I_A \otimes \sigma_\alpha \otimes \mathbb{1}_2, \; \Gamma^i_{(B;\beta)} = I_B \otimes (i\sigma_\beta) \otimes \sigma^i\right), \qquad (157)$$

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The bottom line is:

imposing the Z_3-graded Lorentz invariance on the initial $12-$ *component* generalized Dirac spinor describing a coloured quark state and on the corresponding coloured Dirac equation generates a set of six equivalent representations of this equation. The set of six coloured spinors which splits naturally into three "Lorentz doublets" describes the set of three families (generations) with two flavours each.

According to this model, leptons can be considered as "colourless quarks".

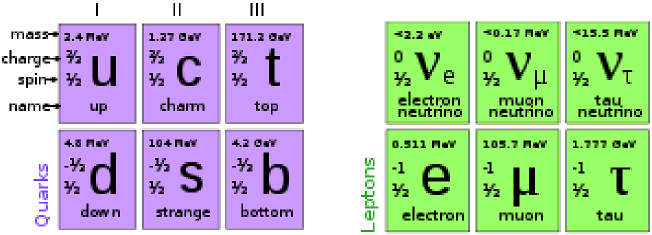

Figure: Three quark generations with two flavours each, and three types of leptons with their neutrinos.

A Z_3-graded analog of Pauli's exclusion principle and the Z_3-graded Dirac's equation were introduced in our papers in 2017, 2018, 2019.

R. Kerner, *Ternary generalization of Pauli's principle and the Z_6-graded algebras,* *Physics of Atomic Nuclei*, **80 (3), pp. 529-531 (2017).**
R. Kerner, *Ternary $Z_2 \times Z_3$ graded algebras and ternary Dirac equation*, *Physics of Atomic Nuclei* **81 (6), pp. 871-889 (2018),**
R. Kerner, *The Quantum nature of Lorentz invariance*, *Universe*, **5 (1), p.1, (2019).**
R. Kerner and J. Lukierski, *Z_3-graded colour Dirac equation for quarks, confinementt and generalized Lorentz symmetries*, *Phys. Letters B*, **Vol. 792, pp. 233-237 (2019),**
R. Kerner and J. Lukierski, *Internal quark symmetries and colour $SU(3)$ entangled with Z_3-graded orentz algebra*, *Nuclear Physics B*, **Vol.** 972, **(November 2021),** 115529